A Course of Mathematics
vol. - 3
1811

PREFACE

TO THE THIRD VOLUME.

THE beneficial improvements lately made, and still making, in the plan of the scientific education of the Cadets, in the Royal Military Academy at Woolwich, having rendered a further extension of the Mathematical Course adviseable, I was honoured with the orders of his Lordship the Master General of the Ordnance, to prepare a third volume, in addition to the two former volumes of the Course, to contain such additions to some of the subjects before treated of in those two volumes, with such other new branches of military science, as might appear best adapted to promote the ends of this important institution. From my advanced age, and the precarious state of my health, I was desirous of declining such a task, and pleaded my doubts of being able, in such a state, to answer satisfactorily his lordship's wishes. This difficulty however was obviated by the reply, that, to preserve a uniformity between the former and the additional parts of the Course, it was requisite that I should undertake the direction of the arrangement, and compose such parts of the work as might be found convenient, or as related to topics in which I had made experiments or improvements; and for the rest,

I might

might take to my assistance the aid of any other person I might think proper. With this kind indulgence being encouraged to exert my best endeavours, I immediately announced my wish to request the assistance of Dr. Gregory of the Royal Military Academy, than whom, both for his extensive scientific knowledge, and his long experience, I know of no person more fit to be associated in the due performance of such a task. Accordingly, this volume is to be considered as the joint composition of that gentleman and myself, having each of us taken and prepared, in nearly equal portions, separate chapters and branches of the work, being such as, in the compass of this volume, with the advice and assistance of the Lieut. Governor, were deemed among the most useful additional subjects for the purposes of the education established in the Academy.

The several parts of the work, and their arrangement, are as follow.—In the first chapter are contained all the propositions of the course of *Conic Sections*, first printed for the use of the Academy in the year 1787, which remained, after those that were selected for the second volume of this Course: to which is added a tract on the algebraic equations of the several conic sections, serving as a brief introduction to the algebraic properties of curve lines.

The 2d chapter contains a short geometrical treatise on the elements of *Isoperimetry* and the *maxima and minima of surfaces and solids*; in which several propositions usually investigated by fluxionary processes are effected geometrically; and in which, indeed, the principal results deduced by Thos. Simpson, Horsley, Legendre, and Lhuillier are thrown into the compass of one short tract.

The 3d and 4th chapters exhibit a concise but comprehensive view of the *trigonometrical analysis*, or that in which the chief theorems of Plane and Spherical Trigonometry are deduced algebraically by means of what is commonly denominated

minated the *Arithmetic of Sines*. A comparison of the modes of investigation adopted in these chapters, and those pursued in that part of the second volume of this course which is devoted to Trigonometry, will enable a student to trace the relative advantages of the algebraical and geometrical methods of treating this useful branch of science. The fourth chapter includes also a disquisition on the nature and measure of *solid angles*, in which the theory of that peculiar class of geometrical magnitudes is so represented, as to render their mutual comparison (a thing hitherto supposed impossible except in one or two very obvious cases) a matter of perfect ease and simplicity.

Chapter the fifth relates to Geodesic Operations, and that more extensive kind of *Trigonometrical Surveying* which is employed with a view to determine the geographical situation of places, the magnitude of kingdoms, and the figure of the earth. This chapter is divided into two sections; in the first of which is presented a general account of this kind of surveying; and in the second, solutions of the most important problems connected with these operations. This portion of the volume it is hoped will be found highly useful; as there is no work which contains a concise and connected account of this kind of surveying and its dependent problems; and it cannot fail to be interesting to those who know how much honour redounds to this country from the great skill, accuracy, and judgment, with which the trigonometrical survey of England has long been carried on.

In the 6th and 7th chapters are developed the principles of *Polygonometry*, and those which relate to the *Division of lands* and other surfaces, both by geometrical construction and by computation.

The 8th chapter contains a view of the nature and solution of *equations* in general, with a selection of the best rules for equations of different degrees. Chapter the 9th is devoted

to

to the nature and properties of *curves*, and the *construction of equations*. These chapters are manifestly connected, and show how the mutual relation subsisting between equations of different degrees, and curves of various orders, serve for the reciprocal illustration of the properties of both.

In the 10th chapter the subjects of *Fluents* and *Fluxional equations* are concisely treated. The various forms of Fluents comprised in the useful table of them in the 2d volume, are investigated: and several other rules are given; such as it is believed will tend much to facilitate the progress of students in this interesting department of science, especially those which relate to the mode of finding fluents by continuation.

The 11th chapter contains solutions of the most useful problems concerning the *maximum effects of machines in motion;* and developes those principles which should constantly be kept in view by those who would labour beneficially for the improvement of machines.

In the 12th chapter will be found the theory of the *pressure of earth and fluids* against walls and fortifications; and the theory which leads to the best construction of *powder magazines* with equilibrated roofs.

The 13th chapter is devoted to that highly interesting subject, as well to the philosopher as to military men, the *theory and practice of gunnery*. Many of the difficulties attending this abstruse enquiry are surmounted by assuming the results of accurate experiments, as to the resistance experienced by bodies moving through the air, as the basis of the computations. Several of the most useful problems are solved by means of this expedient, with a facility scarcely to be expected, and with an accuracy far beyond our most sanguine expectations.

The 14th and last chapter contains a promiscuous but extensive collection of problems in *statics*, *dynamics*, *hydrostatics*, *hydraulics*, *projectiles*, &c, &c; serving at once to

exercise

exercise the pupil in the various branches of mathematics comprised in the course, to demonstrate their utility especially to those devoted to the military profession, to excite a thirst for knowledge, and in several important respects to gratify it.

This volume being professedly supplementary to the preceding two volumes of the Course, may best be used in tuition by a kind of mutual incorporation of its contents with those of the second volume. The method of effecting this will, of course, vary according to circumstances, and the precise employments for which the pupils are destined: but in general it is presumed the following may be advantageously adopted. Let the first seven chapters be taught immediately after the Conic Sections in the 2d volume. Then let the substance of the 2d volume succeed, as far as the Practical Exercises on Natural Philosophy, inclusive. Let the 8th and 9th chapters in this 3d vol. precede the Treatise on Fluxions in the 2d; and when the pupil has been taught the part relating to *fluents* in that treatise, let him immediately be conducted through the 10th chapter of the 3d volume. After he has gone over the remainder of the Fluxions with the applications to tangents, radii of curvature, rectifications, quadratures, &c, the 11th, 12th, and 13th chapters of the 3d vol. should be taught. The problems in the 14th chapter must be blended with the practical exercises at the end of the 2d volume, in such manner as shall be found best suited to the capacity of the student, and best calculated to ensure his thorough comprehension of the several curious problems contained in those portions of the work.

In the composition of this 3d volume, as well as in that of the preceding parts of the Course, the great object kept constantly in view has been *utility*, especially to gentlemen intended for the Military Profession. To this end, all such investigations as might serve merely to display ingenuity or talent, without any regard to practical benefit, have been care-

fully

fully excluded. The student has put into his hands the two powerful instruments of the ancient and the modern or sublime geometry; he is taught the use of both, and their relative advantages are so exhibited as to guard him, it is hoped, from any undue and exclusive preference for either. Much novelty of matter is not to be expected in a work like this; though, considering its magnitude, and the frequency with which several of the subjects have been discussed, a candid reader will not, perhaps, be entirely disappointed in this respect. Perspicuity and condensation have been uniformly aimed at through the performance: and a small clear type, with a full page, have been chosen for the introduction of a large quantity of matter.

A candid public will accept as an apology, for any slight disorder or irregularity, that may appear in the composition and arrangement of this Course, the circumstance of the different volumes having been prepared at widely distant times, and with gradually expanding views. But, on the whole, I trust it will be found that, with the assistance of my friend and coadjutor in this supplementary volume, I have now produced a Course of Mathematics, in which a greater variety of useful subjects are introduced, and treated with perspicuity and correctness, than in any three volumes of equal size in any language whatever.

CHA. HUTTON.

May, 1811.

CONTENTS.

A COURSE OF MATHEMATICS, &c.

CHAPTER I.

CONTINUATION OF THE CONIC SECTIONS.

IN the year 1787 was published, by order of the Master General of the Ordnance, for the use of the Royal Military Academy, a volume of miscellaneous exercises, which had, for many preceding years, been employed in manuscript, in the education of the cadets in the academy. The first and principal article in the contents of that volume, was an extensive geometrical treatise on Conic Sections, treated in a new and a more methodical, as well as easier way, than had been usual.—In the year 1798, when the 2d volume of the Academical Course was first published, by order of the Master General also, the leading propositions of that treatise on Conic Sections were introduced into it.—And now, on the further extension of the Course, by order of his lordship the present Master General, the remaining propositions, of the said first treatise of Conics, are introduced into this 3d volume.

It will be observed that the theorems or propositions in this volume, are numbered in the regular succession from those in the 2d volume, in each of the three sections, commencing here, in the third volume, with the number next following the last in the 2d volume, so as to form these propositions in both

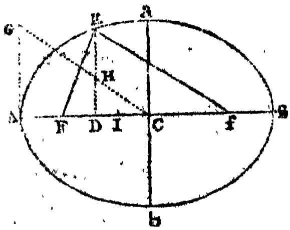
those volumes into a continued series. To each number also will be found annexed, in the Indian figures inclosed in a parenthesis, the number of the same proposition as it was numbered in the first miscellaneous treatise of the year 1787. The additional theorems are as follow.

SECTION I.

OF THE ELLIPSE.

THEOREM XII (5.)

The Difference between the Semi-transverse and a Line drawn from the Focus to any Point in the Curve, is equal to a Fourth Proportional to the Semi-transverse, the Distance from the Centre to the Focus, and the Distance from the Centre to the Ordinate belonging to that Point of the Curve.



That is,
$AC - FE = CI$, or $FE = AI$;
and $fE - AC = CI$, or $fE = BI$.
Where $CA : CF :: CD : CI$ the 4th proportional to CA, CF, CD.

For, draw AG parallel and equal to Ca the semi-conjugate; and join CG meeting the ordinate DE in H.
Then, by theor. 2, $CA^2 : AG^2 :: CA^2 - CD^2 : DE^2$:
and, by sim. tri. $CA^2 : AG^2 :: CA^2 - CD^2 : AG^2 - DH^2$;
consequently $DE^2 = AG^2 - DH^2 = Ca^2 - DH^2$.
Also $FD = CF \sim CD$, and $FD^2 = CF^2 - 2CF . CD + CD^2$;
but by right-angled triangles, $FD^2 + DE^2 = FE^2$;
therefore $FE^2 = CF^2 + Ca^2 - 2CF . CD + CD^2 - DH^2$.
But by theor. 4, $Ca^2 + CF^2 = CA^2$
and, by supposition, $2CF . CD = 2CA . CI$;
theref. $FE^2 = CA^2 - 2CA . CI + CD^2 - DH^2$.
But by supposition, $CA^2 : CD^2 :: CF^2$ or $CA^2 - AG^2 : CI^2$;
and, by sim. tri. $CA^2 : CD^2 :: CA^2 - AG^2 : CD^2 - DH^2$;
therefore - - $CI^2 = CD^2 - DH^2$;
consequently - - $FE^2 = CA^2 - 2CA . CI + CI^2$.
And the root or side of this square is $FE = CA - CI = AI$.
In the same manner is found $fE = CA + CI = BI$. Q. E. D.

Corol.

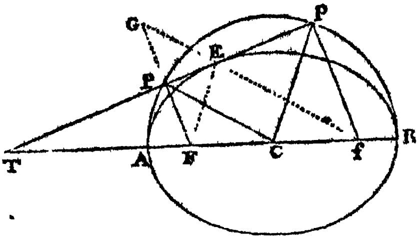
Corol. 1. Hence CI or CA − FE is a 4th proportional to CA, CF, CD.

Corol. 2. And fE − FE = 2CI; that is, the difference between two lines drawn from the foci, to any point in the curve, is double the 4th proportional to CA, CF, CD.

THEOREM XIII (11).

If a Line be drawn from either Focus, Perpendicular to a Tangent to any Point of the Curve; the Distance of their Intersection from the Centre will be equal to the Semi-transverse Axis.

That is, if FP, fp be perpendicular to the tangent TPp, then shall CP and Cp be each equal to CA or CB.



For through the point of contact E draw FE, and fE meeting FP produced in G. Then the ∠GEP = ∠FEP, being each equal to the ∠fEp, and the angles at P being right, and the side PE being common, the two triangles GEP, FEP are equal in all respects, and so GE = FE, and GP = FP. Therefore, since FP = ½FG, and FC = ½Ff, and the angle at F common, the side CP will be = ½fG or ½AB, that is CP = CA or CB. And in the same manner Cp = CA or CB. Q.E.D.

Corol. 1. A circle described on the transverse axis, as a diameter, will pass through the points P, p; because all the lines CA, CP, Cp, CB, being equal, will be radii of the circle.

Corol. 2. CP is parallel to fE, and Cp parallel to FE.

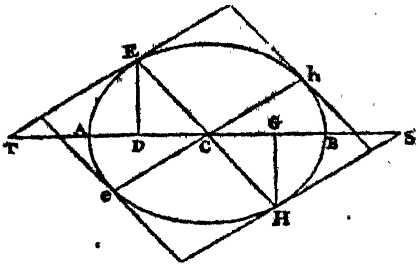
Corol. 3. If at the intersections of any tangent, with the circumscribed circle, perpendiculars to the tangent be drawn, they will meet the transverse axis in the two foci. That is, the perpendiculars PF, pf give the foci F, f.

THEOREM XIV (12).

The equal Ordinates, or the Ordinates at equal Distances from the Centre, on the opposite Sides and Ends of an Ellipse, have their Extremities connected by one Right Line passing through the Centre, and that Line is bisected by the Centre.

That

That is, if CD = CG, or the ordinate DE = GH;
then shall CE = CH, and ECH will be a right line.



For when CD = CG, then also is DE = GH by cor. 2, th. 1.
But the ∠D = ∠G, being both right angles;
therefore the third side CE = CH, and the ∠DCE = ∠GCH,
and consequently ECH is a right line.

Corol. 1. And, conversely, if ECH be a right line passing through the centre; then shall it be bisected by the centre, or have CE = CH; also DE will be = GH, and CD = CG.

Corol. 2. Hence also, if two tangents be drawn to the two ends E, H of any diameter EH; they will be parallel to each other, and will cut the axis at equal angles, and at equal distances from the centre. For, the two CD, CA being equal to the two CG, CB, the third proportionals CT, CS will be equal also; then the two sides CE, CT being equal to the two CH, CS, and the included angle ECT equal to the included angle HCS, all the other corresponding parts are equal: and so the ∠T = ∠S, and TE parallel to HS.

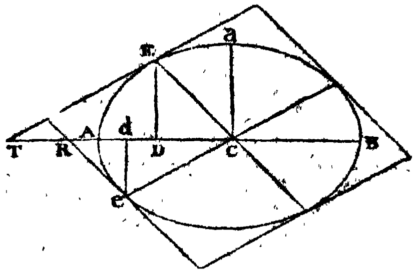
Corol. 3. And hence the four tangents, at the four extremities of any two conjugate diameters, form a parallelogram circumscribing the ellipse, and the pairs of opposite sides are each equal to the corresponding parallel conjugate diameters. For, if the diameter *eh* be drawn parallel to the tangent TE or HS, it will be the conjugate to EH by the definition; and the tangents to *e*, *h* will be parallel to each other, and to the diameter EH for the same reason.

THEOREM XV (13).

If two Ordinates ED, *ed* be drawn from the Extremities E, *e*, of two Conjugate Diameters, and Tangents be drawn to the same Extremities, and meeting the Axis produced in T and R;

Then

Then shall CD be a mean proportional between cd, dR, and cd a mean proportional between CD, DT.



For, by theor. 7, CD : CA :: CA : CT,
and by the same, cd : CA :: CA : CR;
theref. by equality, CD : cd :: CR : CT,
But by sim tri. DT : cd :: CT : CR;
theref. by equality, CD : cd :: cd : DT.
In like manner, cd : CD :: CD : dR. Q.E.D.

Corol. 1. Hence CD : cd :: CR : CT.

Corol. 2. Hence also CD : cd :: de : DE.
And the rectangle CD . DE = cd . de, or Δ CDE = Δ cde.

Corol. 3. Also cd^2 = CD . DT,
and CD^2 = cd . dR,
Or cd a mean proportional between CD, DT;
and CD a mean proportional between cd, dR.

THEOREM XVI (14).

The same Figure being constructed as in the last Theorem, each Ordinate will divide the Axis, and the Semi-axis added to the external Part, in the same Ratio.

[See the last fig.]

That is, DA : DT :: DC : DB,
and dA : dR :: dC : dB.

For, by theor. 7, CD : CA :: CA : CT,
and by div. CD : CA :: AD : AT,
and by comp. CD : DB :: AD : DT,
or, - - - DA : DT :: DC : DB.
In like manner, dA : dR :: dC : dB. Q.E.D.

Corol. 1. Hence, and from cor. 3 to the last, it is,
cd^2 = CD . DT = AD . DB = $\text{CA}^2 - \text{CD}^2$,
CD^2 = cd . dR = Ad . dB = $\text{CA}^2 - cd^2$.

Corol.

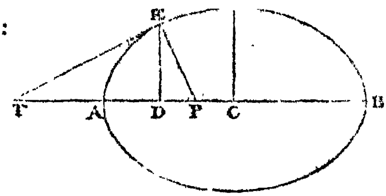
Corol. 2. Hence also, $CA^2 = CD^2 + cd^2$,
and $ca^2 = DE^2 + de^2$.

Corol. 3. Further, because $CA^2 : ca^2 :: AD \cdot DB$ or $cd^2 : DE^2$,
therefore $CA : ca :: cd : DE$,
likewise $CA : ca :: CD : de$.

THEOREM XVII (15).

If from any Point in the Curve there be drawn an Ordinate, and a Perpendicular to the Curve, or to the Tangent at that Point: Then, the

Dist. on the Trans. between the Centre and Ordinate, CD :
Will be to the Dist. PD ::
As Sq. of the Trans. Axis:
To Sq. of the Conjugate.

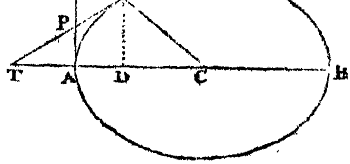


That is,
$CA^2 : ca^2 :: DC : DP$.

For, by theor. 2, $CA^2 : ca^2 :: AD \cdot DB : DE^2$,
But, by rt. angled Δs, the rect. $TD \cdot DP = DE^2$;
and, by cor. 1, theor. 16, $CD \cdot DT = AD \cdot DB$;
therefore - - $CA^2 : ca^2 :: TD \cdot DC : TD \cdot DP$,
or - - - - $AC^2 : ca^2 :: DC : DP$. Q.E.D.

THEOREM XVIII (18).

If there be Two Tangents drawn, the One to the Extremity of the Transverse, and the other to the Extremity of any other Diameter, each meeting the other's Diameter produced; the two Tangential Triangles so formed, will be equal.



That is,
the triangle CET = the triangle CAN.

For, draw the ordinate DE. Then
By sim. triangles, $CD : CA :: CE : CN$;
but, by theor. 7, $CD : CA :: CA : CT$;
theref. by equal. $CA : CT :: CE : CN$.

The

The two triangles CET, CAN have then the angle C common, and the sides about that angle reciprocally proportional; those triangles are therefore equal, namely, the Δ CET $=$ Δ CAN.

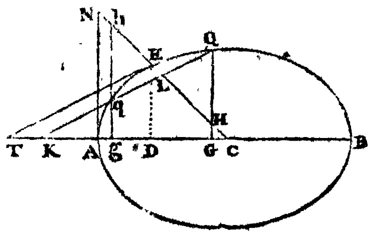
Corol. 1. From each of the equal tri. CET, CAN,
take the common space CAPE,
and there remains the external Δ PAT $=$ Δ PNE.

Corol. 2. Also from the equal triangles CET, CAN,
take the common triangle CED,
and there remains the Δ TED $=$ trapez. ANED.

THEOREM XIX (19).

The same being supposed as in the last Proposition; then any Lines KQ, QG, drawn parallel to the two Tangents, shall also cut off equal Spaces. That is,

ΔKQG $=$ trapez. ANHG,
and ΔKqg $=$ trapez. ANhg.



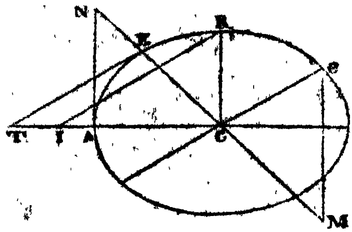
For, draw the ordinate DE. Then
The three sim. triangles CAN, CDE, CGH,
are to each other as CA^2, CD^2, CG^2;
th. by div. the trap. ANED : trap. ANHG :: $CA^2 - CD^2 : CA^2 - CG^2$.
But, by theor. 1, $DE^2 : GQ^2 :: CA^2 - CD^2 : CA^2 - CG^2$.
theref. by equ. trap. ANED : trap. ANHG :: $DE^2 : GQ^2$.
But, by sim. Δs, tri. TED : tri. KQG :: $DE^2 : GQ^2$;
theref. by equality, ANED : TED :: ANHG : KQG.
But, by cor. 2, theor. 18, the trap. ANED $=$ Δ TED;
and therefore the trap. ANHG $=$ Δ KQG.
In like manner the trap. ANhg $=$ Δ Kqg. Q.E.D.

Corol. 1. The three spaces ANHG, TEHG, KQG are all equal.

Corol. 2. From the equals ANHG, KQG,
take the equals ANhg, Kqg,
and there remains ghHG $=$ gqQG.

Corol. 3. And from the equals ghHG, gqQG,
take the common space gqLHG,
and there remains the Δ LQH $=$ ΔLqh.

Corol. 4. Again from the equals KQG, TEHG,
take the common space KLHG,
and there remains TELK $=$ Δ LQH.



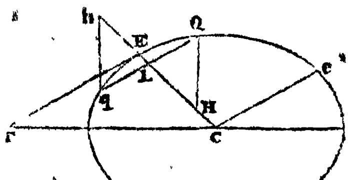
Corol. 5. And when, by the lines KQ, GH, moving with a parallel motion, KQ comes into the position IR, where CR is the conjugate to CA; then

the triangle KQG becomes the triangle IRC,
and the space ANHG becomes the triangle ANC;
and therefore the $\triangle$ IRC $=$ $\triangle$ ANC $=$ $\triangle$ TEC.

Corol. 6. Also when the lines KQ and HQ, by moving with a parallel motion, come into the position C*e*, M*e*,
the triangle LQH becomes the triangle C*e*M,
and the space TELK becomes the triangle TEC;
and theref. the $\triangle$ C*e*M $=$ $\triangle$ TEC $=$ $\triangle$ ANC $=$ $\triangle$ IRC.

THEOREM XX (20).

Any Diameter bisects all its Double Ordinates, or the Lines drawn Parallel to the Tangent at its Vertex, or to its Conjugate Diameter.



That is, if Q*q* be parallel to the tangent TE, or to C*e*, then shall LQ $=$ L*q*.

For, draw QH, *qh* perpendicular to the transverse.
Then by cor. 3, theor. 19, the $\triangle$ LQH $=$ $\triangle$ L*qh*;
but these triangles are also equiangular;
consequently their like sides are equal, or LQ $=$ L*q*.

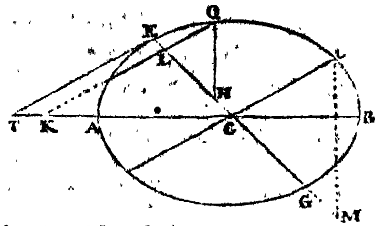
Corol. Any diameter divides the ellipse into two equal parts.

For, the ordinates on each side being equal to each other, and equal in number; all the ordinates, or the area, on one side of the diameter, is equal to all the ordinates, or the area, on the other side of it.

THEOREM XXI (21).

As the Square of any Diameter :
Is to the Square of its Conjugate : :
So is the Rectangle of any two Abscisses :
To the Square of their Ordinate.

That is, $CE^2 : ce^2 :: EL . LG$ or $CE^2 - CL^2 : LQ^2$.



For, draw the tangent TE, and produce the ordinate QL to the transverse at K. Also draw QH, eM perpendicular to the transverse, and meeting EG in H and M.

Then, similar triangles being as the squares of their like sides, it is,

by sim. triangles, $\triangle CET : \triangle CLK :: CE^2 : CL^2$;
or, by division, $\triangle CET :$ trap. $TELK :: CE^2 : CE^2 - CL^2$.
Again, by sim. tri. $\triangle CeM : \triangle LQH :: ce^2 : LQ^2$.
But, by cor. 5 theor. 19, the $\triangle CeM = \triangle CET$,
and, by cor. 4 theor. 19, the $\triangle LQH =$ trap. TELK ;
theref. by equality, $CE^2 : ce^2 :: CE^2 - CL^2 : LQ^2$,
or - - $CE^2 : ce^2 :: EL . LG : LQ^2$. Q.E.D.

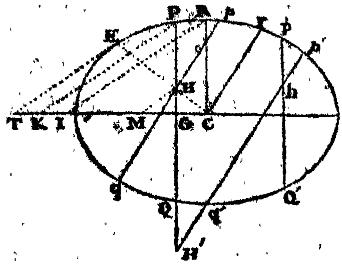
Corol. 1. The squares of the ordinates to any diameter, are to one another as the rectangles of their respective abscisses, or as the difference of the squares of the semidiameter and of the distance between the ordinate and centre. For they are all in the same ratio of CE^2 to ce^2.

Corol. 2. The above being the same property as that belonging to the two axes, all the other properties before laid down, for the axes, may be understood of any two conjugate diameters whatever, using only the oblique ordinates of these diameters, instead of the perpendicular ordinates of the axes ; namely, all the properties in theorems 6, 7, 8, 14, 15, 16, 18 and 19.

THEOREM XXII (22).

If any Two Lines, that any where intersect each other, meet the Curve each in Two Points ; then
The Rectangle of the Segments of the one :
Is to the Rectangle of the Segments of the other : :
As the Square of the Diam. Parallel to the former :
To the Square of the Diam. Parallel to the latter.

That



That is, if CR and cr be Parallel to any two Lines PHQ, pHq: then shall $CR^2 : Cr^2 :: PH . HQ : pH . Hq$.

For, draw the diameter CHE, and the tangent TE, and its parallels PK, RI, MH, meeting the conjugate of the diameter CR in the points T, K, I, M. Then, because similar triangles are as the squares of their like sides, it is,

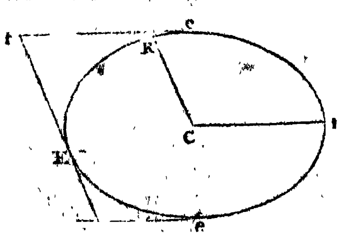
by sim. triangles, $CR^2 : GP^2 :: \Delta CRI : \Delta GPK$,
and - - - - $CR^2 : GH^2 :: \Delta CRI : \Delta GHM$;
theref. by division, $CR^2 : GP^2 - GH^2 :: CRI : KPHM$.
Again, by sim. tri. $CE^2 : CH^2 :: \Delta CTE : \Delta CMH$;
and by division, $CE^2 : CE^2 - CH^2 :: \Delta CTE : TEHM$.
But, by cor. 5 theor. 19, the $\Delta CTE = \Delta CIR$,
and by cor. 1 theor. 19, $TEHG = KPHG$, or $TEHM = KPHM$;
theref. by equ. $CE^2 : CE^2 - CH^2 :: CR^2 : GP^2 - GH^2$ or $PH . HQ$.
In like manner $CE^2 : CE^2 - CH^2 :: Cr^2 : pH . Hq$.
Theref. by equ. $CR^2 : Cr^2 :: PH . HQ : pH . Hq$. Q.E.D.

Corol. 1. In like manner, if any other line $p'H'q'$, parallel to Cr or to pq, meet PHQ; since the rectangles $PH'Q$, $p'H'q'$ are also in the same ratio of CR^2 to Cr^2; therefore rect. $PHQ : pHq :: PH'Q : p'H'q'$.

Also, if another line $P'hQ'$ be drawn parallel to PQ or CR; because the rectangles $P'hQ'$, $p'hq'$ are still in the same ratio, therefore, in general, the rect. $PHQ : pHq :: P'hQ' : p'hq'$.

That is, the rectangles of the parts of two parallel lines, are to one another, as the rectangles of the parts of two other parallel lines, any where intersecting the former.

Corol. 2. And when any of the lines only touch the curve, instead of cutting it, the rectangles of such become squares, and the general property still attends them.



That is,
$CR^2 : Cr^2 :: TE^2 : Te^2$,
or $CR : Cr :: TE : Te$.
and $CR : Cr :: tE : te$.

Corol. 3. And hence $TE : Te :: tE : te$.

SECTION

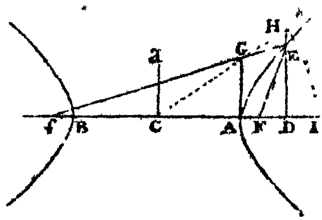
SECTION II.

OF THE HYPERBOLA.

THEOREM XIV (5).

The Sum or Difference of the Semi-transverse and a Line drawn from the Focus to any Point in the Curve, is equal to a Fourth Proportional to the Semi-transverse, the Distance from the Centre to the Focus, and the Distance from the Centre to the Ordinate belonging to that Point of the Curve.

That is,
$FE + AC = CI$, or $FE = AI$;
and $fE - AC = CI$, or $fE = BI$.
Where $CA : CF :: CD : CI$ the 4th propor. to CA, CF, CD.



For, draw AG parallel and equal to Ca the semi-conjugate; and join CG meeting the ordinate DE produced in H.

Then, by theor. 2, $CA^2 : AG^2 :: CD^2 - CA^2 : DE^2$;
and, by sim. Δs, $CA^2 : AG^2 :: CD^2 - CA^2 : DH^2 - AG^2$;
consequently $DE^2 = DH^2 - AG^2 = DH^2 - Ca^2$.

Also $FD = CF \sim CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
but, by right angled triangles, $FD^2 + DE^2 = FE^2$;
therefore $FE^2 = CF^2 - Ca^2 - 2CF \cdot CD + CD^2 + DH^2$.

But by theor. 4, $CF^2 - Ca^2 = CA^2$,
and, by supposition, $2CF \cdot CD = 2CA \cdot CI$;
theref. $FE^2 = CA^2 - 2CA \cdot CI + CD^2 + DH^2$.

But, by supposition, $CA^2 : CD^2 :: CF^2$ or $CA^2 + AG^2 : CI^2$;
and, by sim. Δs, $CA^2 : CD^2 :: CA^2 + AG^2 : CD^2 + DH^2$;
therefore - $CI^2 = CD^2 + DH^2 = CH^2$;
consequently - $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.
And the root or side of this square is $FE = CI - CA = AI$.
In the same manner is found $fE = CI + CA = BI$. Q.E.D.

Corol. 1. Hence $CH = CI$ is a 4th propor. to CA, CF, CD.

Corol.

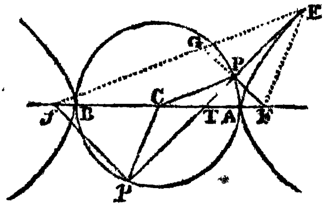
Corol. 2. And fE + FE = 2CH or 2CI; or FE, CH, fE are in continued arithmetical progression, the common difference being CA the semi-transverse.

Corol. 3. From the demonstration it appears, that $DE^2 = DH^2 - AG^2 = DH^2 - Ca^2$. Consequently DH is every where greater than DE; and so the asymptote CGH never meets the curve, though they be ever so far produced: but DH and DE approach nearer and nearer to a ratio of equality as they recede farther from the vertex, till at an infinite distance they become equal, and the asymptote is a tangent to the curve at an infinite distance from the vertex.

THEOREM XV (11).

If a Line be drawn from either Focus, Perpendicular to a Tangent to any Point of the Curve; the Distance of their Intersection from the Centre will be equal to the Semi-transverse Axis.

That is, if FP, fp be perpendicular to the tangent TPp, then shall CP and Cp be each equal to CA or CB.



For, through the point of contact E draw FE, and fE, meeting FP produced in G. Then, the ∠ GEP = ∠ FEP, being each equal to the ∠ fEP, and the angles at P being right, and the side PE being common, the two triangles GEP, FEP are equal in all respects, and so GE = FE, and GP = FP. Therefore, since FP = $\frac{1}{2}$FG, and FC = $\frac{1}{2}$Ff, and the angle at F common, the side CP will be = $\frac{1}{2}f$G or $\frac{1}{2}$AB, that is CP = CA or CB.

And in the same manner Cp = CA or CB. Q.E.D.

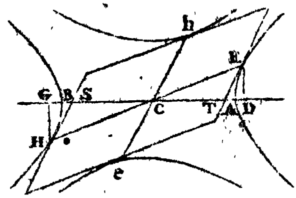
Corol. 1. A circle described on the transverse axis, as a diameter, will pass through the points P, p; because all the lines CA, CP, Cp, CB, being equal, will be radii of the circle.

Corol. 2. CP is parallel to fE, and Cp parallel to FE.

Corol. 3. If at the intersections of any tangent, with the circumscribed circle, perpendiculars to the tangent be drawn, they will meet the transverse axis in the two foci. That is the perpendiculars PF, pf give the foci F, f.

THEOREM XVI (12).

The equal Ordinates, or the Ordinates at equal Distances from the Centre, on the opposite Sides and Ends of an Hyperbola, have their Extremities connected by one Right Line passing through the Centre, and that Line is bisected by the Centre.



That is, if CD = CG, or the ordinate DE = GH; then shall CE = CH, and ECH will be a right line.

For, when CD = CG, then also is DE = GH by cor. 2 theor. 1. But the ∠D = ∠G, being both right angles; therefore the third side CE = CH, and the ∠DCE = ∠GCH, and consequently ECH is a right line.

Corol. 1. And, conversely, if ECH be a right line passing through the centre; then shall it be bisected by the centre, or have CE = CH, also DE will be = GH, and CD = CG.

Corol. 2. Hence also, if two tangents be drawn to the two ends E, H of any diameter EH; they will be parallel to each other, and will cut the axis at equal angles, and at equal distances from the centre. For, the two CD, CA being equal to the two CG, CB, the third proportionals CT, CS will be equal also; then the two sides CE, CT being equal to the two CH, CS, and the included angle ECT equal to the included angle HCS, all the other corresponding parts are equal: and so the ∠T = ∠S, and TE parallel to HS.

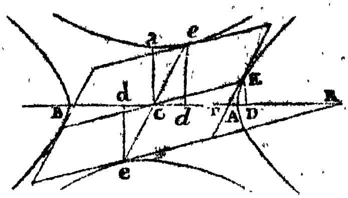
Corol. 3. And hence the four tangents, at the four extremities of any two conjugate diameters, form a parallelogram inscribed between the hyperbolas, and the pairs of opposite sides are each equal to the corresponding parallel conjugate diameters.—For, if the diameter *eh* be drawn parallel to the tangent TE or HS, it will be the conjugate to EH by the definition; and the tangents to *e*, *h* will be parallel to each other, and to the diameter EH for the same reason.

THEOREM XVII (13).

If two Ordinates ED, *ed* be drawn from the Extremities E, *e*, of two Conjugate Diameters, and Tangents be drawn to the same Extremities, and meeting the Axis produced in T and R;

Then

Then shall CD be a mean proportional between cd, dR,
and cd a mean proportional between CD, DT.



For, by theor. 7, CD : CA : : CA : CT,
and by the same, cd : CA : : CA : CR;
theref. by equality, CD : cd : : CR : CT.
But by sim. tri. DT : cd : : CT : CR;
theref. by equality, CD : cd : : cd : DT.
In like manner, cd : CD : : CD : dR. Q.E.D.

Corol. 1. Hence CD : cd : : CR : CT.

Corol. 2. Hence also CD : cd : : de : DE.
And the rect. CD . DE $=$ cd . de, or $\triangle$ CDE $=$ $\triangle$ cde.

Corol. 3. Also $cd^2 =$ CD . DT, and CD$^2 = cd$. dR.
Or cd a mean proportional between CD, DT;
and CD a mean proportional between cd, dR.

THEOREM XVIII (14).

The same Figure being constructed as in the last Proposition, each Ordinate will divide the Axis, and the Semi-axis added to the external Part, in the same Ratio.

[See the last fig.]

That is, DA : DT : : DC : DB,
and dA : dR : : dc : dB.

For, by theor. 7, CD : CA : : CA : CT,
and by div. CD : CA : : AD : AT,
and by comp. CD : DB : : AD : DT,
or - - DA : DT : : DC : DB.
In like manner, dA : dR : : dc : dB. Q.E.D.

Corol. 1. Hence, and from cor. 3 to the last prop. it is,
$cd^2 =$ CD . DT $=$ AD . DB $=$ CD$^2 -$ CA^2,
and CD$^2 = cd$. dR $=$ Ad . dB $=$ CA$^2 - cd^2$.

Corol. 2. Hence also CA$^2 =$ CD$^2 - cd^2$, and $ca^2 = de^2 -$ DE^2.

Corol. 3. Farther, because CA^2 : ca^2 : : AD . DB or cd^2 : DE^2.
therefore CA : ca : : cd : DE.
likewise CA : ca : : CD : de.

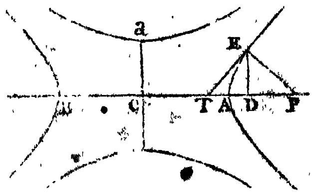
THEOREM

THEOREM XIX (15).

If from any Point in the Curve there be drawn an Ordinate, and a Perpendicular to the Curve, or to the Tangent at that Point: Then the

Dist. on the Trans. between the Centre and Ordinate, CD:
Will be to the Dist. PD ::
As Square of Trans. Axis :
To Square of the Conjugate.

That is,
$CA^2 : Ca^2 :: DC : DP$.

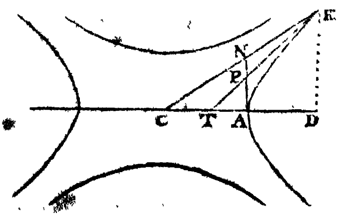


For, by theor. 2, $CA^2 : Ca^2 :: AD \cdot DB : DE^2$,
But, by rt. angled Δs, the rect. $TD \cdot DP = DE^2$;*
and, by cor. 1 theor. 16, $CD \cdot DT = AD \cdot DB$;
therefore - - $CA^2 : Ca^2 :: TD \cdot DC : TD \cdot DP$,
or - - - - - $CA^2 : Ca^2 :: DC : DP$. Q.E.D.

THEOREM XX (18).

If there be Two Tangents drawn, the One to the Extremity of the Transverse, and the other to the Extremity of any other Diameter, each meeting the other's Diameter produced: the two Tangential Triangles so formed, will be equal.

That is,
the triangle CET =
the triangle CAN



For, draw the ordinate DE. Then
By sim. triangles, $CD : CA :: CE : CN$;
but, by theor. 7, $CD : CA :: CA : CT$;
theref. by equal. $CA : CT :: CE : CN$.

The two triangles CET, CAN have then the angle C common, and the sides about that angle reciprocally proportional; those triangles are therefore equal, viz. the $\Delta CET = \Delta CAN$. Q.E.D.

Corol.

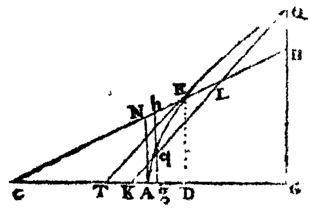
Corol. 1. Take each of the equal tri. CET, CAN,
from the common space CAPE,
and there remains the external Δ PAT = Δ PNE.

Corol. 2. Also take the equal triangles CET, CAN,
from the common triangle CED,
and there remains the Δ TED = trapez. ANED.

THEOREM XXI (19).

The same being supposed as in the last Proposition; then any Lines KQ, GQ, drawn parallel to the two Tangents, shall also cut off equal Spaces.

That is,
the Δ KQG = trapez. ANHG.
and Δ Kqg = trapez. ANhg.



For, draw the ordinate DE. Then
The three sim. triangles CAN, CDE, CGH,
are to each other as CA^2, CD^2, CG^2;
th. by div. the trap. ANED : trap. ANHG :: CD^2 — CA^2 : CG^2 — CA^2.
But, by theor. 1, DE^2 : GQ^2 :: CD^2 — CA^2 : CG^2 — CA^2;
theref. by equ. trap. ANED : trap. ANHG :: DE^2 : GQ^2.
But, by sim. Δs, tri. TED : tri. KQG :: DE^2 : GQ^2;
theref. by equal. ANED : TED :: ANHG : KQG.
But, by cor. 2 theor. 20, the trap. ANED = Δ TED;
and therefore the trap. ANHG = Δ KQG.
In like manner the trap. ANhg = Δ Kqg. Q.E.D.

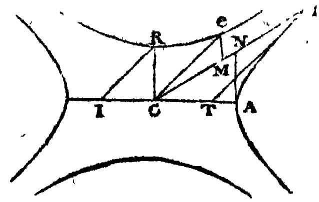
Corol. 1. The three spaces ANHG, TEHG, KQG are all equal.

Corol. 2. From the equals ANHG, KQG,
take the equals ANhg, Kqg,
and there remains ghHG = gqQG.

Corol. 3. And from the equals ghHG, gqQG,
take the common space gqLHG,
and there remains the Δ LQH = Δ Lqh.

Corol. 4. Again, from the equals KQG, TEHG,
take the common space KLHG,
and there remains TELK = Δ LQH.

Corol.



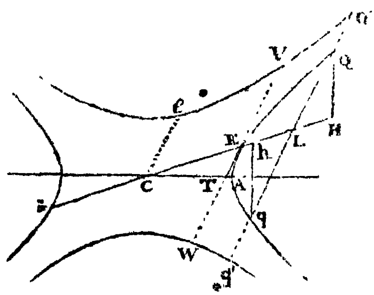
Corol. 5. And when, by the lines KQ, GH, moving with a parallel motion, KQ comes into the position IR, where CR is the conjugate to CA; then

the triangle KQG becomes the triangle IRC,
and the space ANHG becomes the triangle ANC;
and therefore the Δ IRC = Δ ANC = Δ TEC.

Corol. 6. Also when the lines KQ and HQ, by moving with a parallel motion, come into the position C*e*, M*e*,
the triangle LQH becomes the triangle C*e*M,
and the space TELK becomes the triangle TEC;
and theref. the Δ C*e*M = Δ TEC = Δ ANC = Δ IRC.

THEOREM XXII (20).

Any Diameter bisects all its Double Ordinates, or the Lines drawn Parallel to the Tangent at its Vertex, or to its Conjugate Diameter.



That is, if Q*q* be parallel to the tangent TE, or to C*e*, then shall LQ = L*q*.

For, draw QH, *qh* perpendicular to the transverse.
Then by cor. 3 theor. 21, the Δ LQH = Δ L*qh*;
but these triangles are also equiangular;
conseq. their like sides are equal, or LQ = L*q*.

Corol. 1. Any diameter divides the hyperbola into two equal parts.

For, the ordinates on each side being equal to each other, and equal in number; all the ordinates, or the area, on one side of the diameter, is equal to all the ordinates, or the area, on the other side of it.

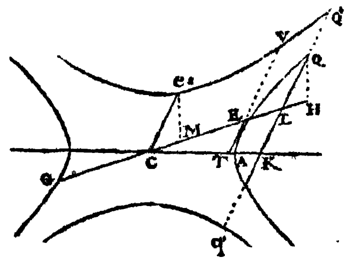
Corol. 2. In like manner, if the ordinate be produced to the conjugate hyperbolas at Q′, *q*′, it may be proved that

$LQ' = Lq'$. Or if the tangent TE be produced, then $EV = EW$. Also the diameter GCEH bisects all lines drawn parallel to TE or Qq, and limited either by one hyperbola, or by its two conjugate hyperbolas.

THEOREM XXIII (21).

As the Square of any Diameter :
Is to the Square of its Conjugate : :
So is the Rectangle of any two Abscisses :
To the Square of their Ordinate.

That is, $CE^2 : ce^2 :: EL . LG$ or $CL^2 - CE^2 : LQ^2$.



For, draw the tangent TE, and produce the ordinate QL to the transverse at K. Also draw QH, eM perpendicular to the transverse, and meeting EG in H and M. Then, similar triangles being as the squares of their like sides, it is,

by sim. triangles, $\Delta CET : \Delta CLK :: CE^2 : CL^2$;
or, by division, ΔCET : trap. $TELK :: CE^2 : CL^2 - CE^2$.
Again, by sim. tri. $\Delta CeM : \Delta LQH :: ce^2 : LQ^2$.
But, by cor. 5 theor. 21, the $\Delta CeM = \Delta CET$,
and, by cor. 4 theor. 21, the $\Delta LQH =$ trap. TELK;
theref. by equality, $CE^2 : ce^2 :: CL^2 - CE^2 : LQ^2$,
or - - - $CE^2 : ce^2 :: EL . LG : LQ^2$. Q.E.D.

Corol. 1. The squares of the ordinates to any diameter, are to one another as the rectangles of their respective abscisses, or as the difference of the squares of the semi-diameter and of the distance between the ordinate and centre. For they are all in the same ratio of CE^2 to ce^2.

Corol. 2. The above being the same property as that belonging to the two axes, all the other properties before laid down, for the axes, may be understood of any two conjugate diameters whatever, using only the oblique ordinates of these diameters instead of the perpendicular ordinates of the axes; namely, all the properties in theorems 6, 7, 8, 16, 17, 20, 21.

Corol. 3. Likewise, when the ordinates are continued to the conjugate hyperbolas at Q', q', the same properties still obtain, substituting only the sum for the difference of the squares of CE and CL,

That is, $CE^2 : ce^2 :: CL^2 + CE^2 : LQ'^2$.
And so $LQ^2 : LQ'^2 :: CL^2 - CE^2 : CL^2 + CE^2$.

Corol.

Corol. 4. When, by the motion of LQ′ parallel to itself, that line coincides with EV, the last corollary becomes

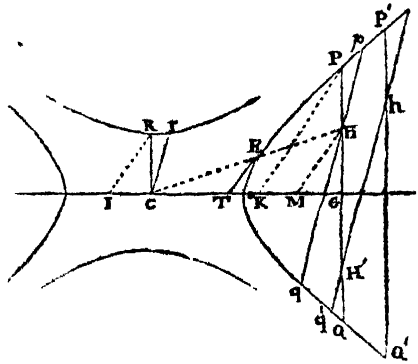
$$\text{CE}^2 : ce^2 :: 2\text{CE}^2 : \text{EV}^2,$$
$$\text{or } ce^2 : \text{EV}^2 :: 1 : 2,$$
$$\text{or } ce : \text{EV} :: 1 : \sqrt{2},$$

or as the side of a square to its diagonal.

That is, in all conjugate hyperbolas, and all their diameters, any diameter is to its parallel tangent, in the constant ratio of the side of a square to its diagonal.

THEOREM XXIV (22).

If any Two Lines, that any where intersect each other, meet the Curve each in Two Points; then
The Rectangle of the Segments of the one :
Is to the Rectangle of the Segments of the other : :
As the Square of the Diam. Parallel to the former :
To the Square of the Diam. Parallel to the latter.



That is, if CR and cr be parallel to any two lines PHQ, pHq; then shall $\text{CR}^2 : cr^2 :: \text{PH}\cdot\text{HQ} : p\text{H}\cdot\text{H}q$.

For, draw the diameter CHE, and the tangent TE, and its parallels PK, RI, MH, meeting the conjugate of the diameter CR in the points T, K, I, M. Then, because similar triangles are as the squares of their like sides, it is,

by sim. triangles, $\text{CR}^2 : \text{GP}^2 :: \Delta\,\text{CRI} : \Delta\,\text{GPK}$,
and - - $\text{CR}^2 : \text{GH}^2 :: \Delta\,\text{CRI} : \Delta\,\text{GHM}$;
theref. by division, $\text{CR}^2 : \text{GP}^2 - \text{GH}^2 :: \text{CRI} : \text{KPHM}$.
Again, by sim. tri. $\text{CE}^2 : \text{CH}^2 :: \Delta\,\text{CTE} : \Delta\,\text{CMH}$;
and by division, $\text{CE}^2 : \text{CH}^2 - \text{CE}^2 :: \Delta\,\text{CTE} : \text{TEHM}$.

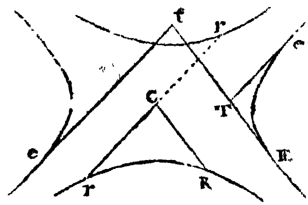
But, by cor. 5 theor. 21, the $\Delta\,\text{CTE} = \Delta\,\text{CIR}$,
and by cor. 1 theor. 21, $\text{TEHG} = \text{KPHG}$, or $\text{TEHM} = \text{KPHM}$;
theref. by equ. $\text{CE}^2 : \text{CH}^2 - \text{CE}^2 :: \text{CR}^2 : \text{GP}^2 - \text{GH}^2$ or PH . HQ.
In like manner $\text{CE}^2 : \text{CH}^2 - \text{CE}^2 :: cr^2 : p\text{H}\cdot\text{H}q$.
Theref. by equ. $\text{CR}^2 : cr^2 :: \text{PH}\cdot\text{HQ} : p\text{H}\cdot\text{H}q$. Q.E.D.

Corol. 1. In like manner, if any other line $p'H'q'$, parallel to cr or to pq, meet PHQ; since the rectangles PH'Q, $p'H'q'$ are also in the same ratio of CR^2 to cr^2; therefore the rect. PHQ : pHq : : PH'Q : $p'H'q'$.

Also, if another line $P'hQ'$ be drawn parallel to PQ or CR; because the rectangles $P'hQ'$, $p'hq'$ are still in the same ratio, therefore, in general, the rectangle PHQ : pHq : : $P'hQ'$: $p'hq'$. That is, the rectangles of the parts of two parallel lines, are to one another, as the rectangles of the parts of two other parallel lines, any where intersecting the former.

Corol. 2. And when any of the lines only touch the curve, instead of cutting it, the rectangles of such become squares, and the general property still attends them.

That is,
$CR^2 : cr^2 :: TE^2 : Te^2$,
or $CR : cr :: TE : Te$,
and $CR : cr :: tE : te$,

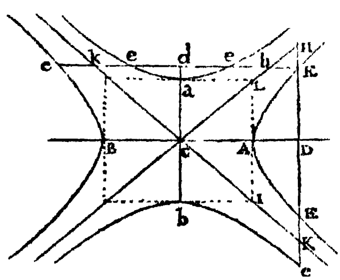


Corol. 3. And hence TE $Te :: tE : te$.

THEOREM XXV (23).

If a Line be drawn through any Point of the Curves, Parallel to either of the Axes, and terminated at the Asymptotes; the Rectangle of its Segments, measured from that Point, will be equal to the Square of the Semi-axis to which it is parallel.

That is,
the rect. HEK or $HeK = Ca^2$,
and rect. hEk or $hek = CA^2$.



For, draw AL parallel to ca, and aL to CA. Then by the parallels, $CA^2 : ca^2$ or $AL^2 :: CD^2 : DH^2$;
and, by theor. 2, $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$;
theref. by subtr. $CA^2 : ca^2 :: CA^2 : DH^2 - DE^2$ or HEK.
But the antecedents CA^2, CA^2 are equal,
theref. the consequents ca^2, HEK must also be equal

In

In like manner it is again,
by the parallels, $CA^2 : ca^2$ or $AL^2 :: CD^2 : DH^2$;
and, by theor. 3, $CA^2 : ca^2 :: CD^2 + CA^2 : De^2$;
theref. by subtr. $CA^2 : ca^2 :: CA^2 : De^2 - DH^2$ or HeK.
But the antecedents CA^2, CA^2 are the same,
theref. the conseq. ca^2, eK must be equal.
In like manner, by changing the axes, is hEk or $hek = CA^2$.

Corol. 1. Because the rect HEK = the rect. HeK.
therefore $EH : eH :: eK : EK$.
And consequently HE is always greater than He.

Corol. 2. The rectangle hEK = the rect. HEk.
For, by sim. tri. $Eh : EH :: Ek : EK$.

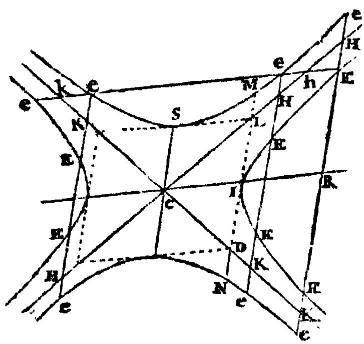
SCHOLIUM.

It is evident that this proposition is general for any line oblique to the axis also, namely, that the rectangle of the segments of any line, cut by the curve, and terminated by the asymptotes, is equal to the square of the semi-diameter to which the line is parallel. Since the demonstration is drawn from properties that are common to all diameters.

THEOREM XXVI (24).

All the Rectangles are equal which are made of the Segments of any Parallel Lines cut by the Curve, and limited by the Asymptotes.

That is,
the rect. $HEK = HeK$.
and rect. $hEk = hek$.



For, each of the rectangles HEK or HeK is equal to the square of the parallel semi-diameter CS; and each of the rectangles hEk or hek is equal to the square of the parallel semi-diameter CI. And therefore the rectangles of the segments of all parallel lines are equal to one another. Q.E.D.

Corol.

Corol. 1. The rectangle HEK being constantly the same, whether the point E is taken on the one side or the other of the point of contact I of the tangent parallel to HK, it follows that the parts HE, KE, of any line HK, are equal.

And because the rectangle H*e*K is constant, whether the point *e* is taken in the one or the other of the opposite hyperbolas, it follows, that the parts H*e*, K*e*, are also equal.

Corol. 2. And when HK comes into the position of the tangent DIL, the last corollary becomes IL = ID, and IM = IN, and LM = DN.

Hence also the diameter CIR bisects all the parallels to DL which are terminated by the asymptote, namely RH = RK.

Corol. 3. From the proposition, and the last corollary, it follows that the constant rectangle HEK or EHE is $= IL^2$. And the equal constant rect. H*e*K or *e*H*e* = MLN or $IM^2 - IL^2$.

Corol. 4. And hence IL = the parallel semi-diameter CS.

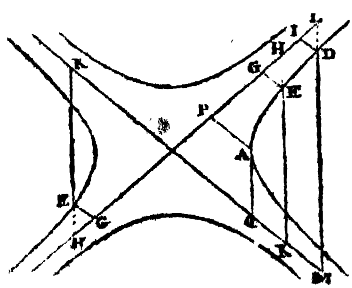
For, the rect. $EHE = IL^2$;
and the equal rect. $eHe = IM^2 - IL^2$,
theref. $IL^2 = IM^2 - IL^2$, or $IM^2 = 2IL^2$;
but, by cor. 4 theor. 23, $IM^2 = 2CS^2$,
and therefore - - $IL = CS$.

And so the asymptotes pass through the opposite angles of all the inscribed parallelograms.

THEOREM XXVII (25).

The Rectangle of any two Lines drawn from any Point in the Curve, Parallel to two given Lines, and Limited by the Asymptotes, is a Constant Quantity.

That is, if AP, EG, DI be parallels,
as also AQ, EK, DM parallels,
then shall the rect. PAQ = rect. GEK = rect. IDM.



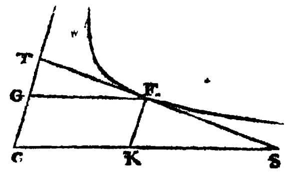
For,

For, produce KE, MD to the other asymptote at H, L.
Then, by the parallels, HE : GE :: LD : ID;
but - - - EK : EK :: DM : DM;
theref. the rectangle HEK : GEK :: LDM : IDM.
But, by the last theor. the rect. HEK = LDM;
and therefore the rect. GEK = IDM = PAQ. Q.E.D.

THEOREM XXVIII (27).

Every Inscribed Triangle, formed by any Tangent and the two Intercepted Parts of the Asymptotes, is equal to a Constant Quantity; namely Double the Inscribed Parallelogram.

That is, the triangle CTS = 2 paral. GK.

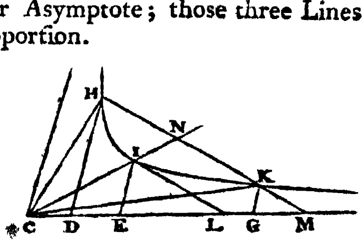


For, since the tangent TS is bisected by the point of contact E, and EK is parallel to TC, and GE to CK; therefore CK, KS, GE are all equal, as are also CG, GT, KE. Consequently the triangle GTE = the triangle KES, and each equal to half the constant inscribed parallelogram GK. And therefore the whole triangle CTS, which is composed of the two smaller triangles and the parallelogram, is equal to double the constant inscribed parallelogram GK. Q.E.D.

THEOREM XXIX (29).

If from the Point of Contact of any Tangent, and the two Intersections of the Curve with a Line parallel to the Tangent, three parallel Lines be drawn in any Direction, and terminated by either Asymptote; those three Lines shall be in continued Proportion.

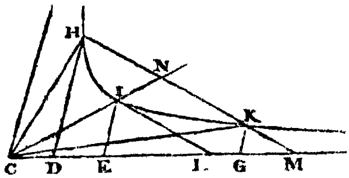
That is, if HKM and the tangent IL be parallel, then are the parallels DH, EI, GK in continued proportion.



For, by the parallels, EI : IL :: DH : HM;
and, by the same, EI : IL :: GK : KM;
theref. by compos. $EI^2 : IL^2$ :: DH . GK : HMK;
but, by theor. 26, the rect. HMK = IL^2;
and theref. the rect. DH . GK = EI^2,
or - - - DH : EI :: EI : GK. Q.E.D.

THEOREM XXX (30).

Draw the semi-diameters CH, CIN, CK;
Then shall the sector CHI = the sector CIK.



For, because HK and all its parallels are bisected by CIN,
therefore the triangle CNH = tri. CNK,
and the segment INH = seg. INK;
consequently the sector CIH = sec. CIK.

Corol. If the geometricals DH, EI, GK be parallel to the other asymptote, the spaces DHIE, EIKG will be equal; for they are equal to the equal sectors CHI, CIK.

So that by taking any geometricals CD, CE, CG, &c, and drawing DH, EI, GK, &c, parallel to the other asymptote, as also the radii CH, CI, CK;
then the sectors CHI, CIK, &c,
or the spaces DHIE, EIKG, &c,
will be all equal among themselves.
Or the sectors CHI, CHK, &c,
or the spaces DHIE, DHKG, &c,
will be in arithmetical progression.

And therefore these sectors, or spaces, will be analogous to the logarithms of the lines or bases CD, CE, CG, &c; namely CHI or DHIE the log. of the ratio of
CD to CE, or of CE to CG, &c; or of EI to DH, or of GK to EI, &c;
and CHK or DHKG the log. of the ratio of
CD to CG, &c, or of GK to DH, &c.

SECTION III.

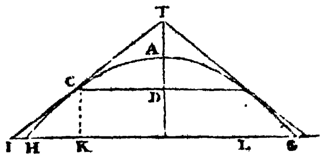
OF THE PARABOLA.

THEOREM XX (7).

If an Ordinate be drawn to the Point of Contact of any Tangent, and another Ordinate produced to cut the Tangent; It will be, as the Difference of the Ordinates:
Is to the Difference added to the external Part::
So is Double the first Ordinate:
To the Sum of the Ordinates.

That

That is, KH : KI :: KL : KG.



For, by cor. 1 theor. 1, P : DC :: DC : DA,
and - - - P : 2DC :: DC : DT or 2DA.
But, by sim. triangles, KI : KC :: DC : DT;
therefore by equality, P : 2DC :: KI : KC,
or, - - - P : KI :: KL : KC.
Again, by theor. 2, P : KH :: KG : KC;
therefore by equality, KH : KI :: KL : KG. Q.E.D.

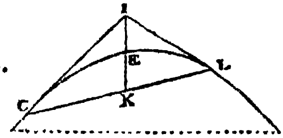
Corol. 1. Hence, by composition and division,
it is, KH : KI :: GK : GI,
and HI : HK :: HK : KL,
also IH : IK :: IK : IG;
that is, IK is a mean proportional between IG and IH.

Corol. 2. And from this last property a tangent can easily be drawn to the curve from any given point I. Namely, draw IHG perpendicular to the axis, and take IK a mean proportional between IH, IG; then draw KC parallel to the axis, and C will be the point of contact, through which and the given point I the tangent IC is to be drawn.

THEOREM XXI (16).

If a Tangent cut any Diameter produced, and if an Ordinate to that Diameter be drawn from the Point of Contact; then the Distance in the Diameter produced, between the Vertex and the Intersection of the Tangent, will be equal to the Absciss of that Ordinate.

That is, IE = EK.
For, by the last th. IE : EK :: CK : KL.
But, by theor. 11, CK = KL,
and therefore IE = EK.



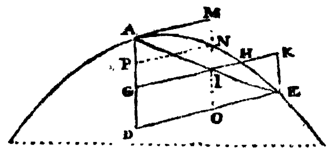
Corol. 1. The two tangents CI, LI, at the extremities of any double ordinate CL, meet in the same point of the diameter of that double ordinate produced. And the diameter drawn through the intersection of two tangents, bisects the line connecting the points of contact.

Corol.

Corol. 2. Hence we have another method of drawing a tangent from any given point I without the curve. Namely, from I draw the diameter IK, in which take EK = EI, and through K draw CL parallel to the tangent at E; then C and L are the points to which the tangents must be drawn from I.

THEOREM XXII (18).

If a Line be drawn from the Vertex of any Diameter, to cut the Curve in some other Point, and an Ordinate of that Diameter be drawn to that Point, as also another Ordinate any where cutting the Line, both produced if necessary: The Three will be continual Proportionals, namely, the two Ordinates and the Part of the Latter limited by the said Line drawn from the Vertex.



That is, DE, GH, GI are continual proportionals, or DE : GH : : GH : GI.

For, by theor. 9, - - -	$DE^2 : GH^2 :: AD : AG$;	
and, by sim. tri. - - -	$DE : GI :: AD : AG$;	
theref. by equality, - -	$DE : GI :: DE^2 : GH^2$,	
that is, of the three DE, GH, GI,	1st : 3d :: 1st² : 2d²;	
therefore - - - - -	1st : 2d :: 2d : 3d,	
that is, - - - - - - -	DE : GH :: GH : GI.	Q.E.D.

Corol. 1. Or their equals, GK, GH, GI, are proportionals; where EK is parallel to the diameter AD.

Corol. 2. Hence it is DE : AG :: p : GI, where p is the parameter, or AG : GI :: DE : p.
For, by the defin. AG : GH :: GH : p.

Corol. 3. Hence also the three MN, MI, MO, are proportionals, where MO is parallel to the diameter, and AM parallel to the ordinates.

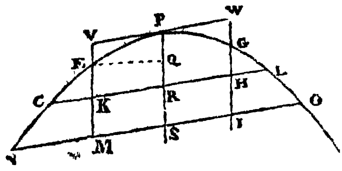
For, by theor. 9, -	MN, MI, MO,
or their equals -	AP, AG, AD,
are as the squares of	PN, GH, DE,
or of their equals	GI, GH, GK,
which are proportionals by cor. 1.	

THEOREM XXIII (19).

If a Diameter cut any Parallel Lines terminated by the Curve; the Segments of the Diameter will be as the Rectangle of the Segments of those Lines.

That

That is, EK : EM :: CK . KL : NM . MO.
Or, EK is as the rectangle CK . KL.



For, draw the diameter PS to which the parallels CL, NO are ordinates, and the ordinate EQ parallel to them.

Then CK is the difference, and KL the sum of the ordinates EQ, CR; also NM the difference, and MO the sum of the ordinates EQ, NS. And the differences of the abscisses, are QR, QS, or EK, EM.

Then by cor. theor. 9, QR : QS :: CK . KL : NM . MO,
that is - - EK : EM :: CK . KL : NM . MO.

Corol. 1. The rect. CK.KL = rect. EK and the param. of PS.
For the rect. CK.KL = rect. QR and the param. of PS.

Corol. 2. If any line CL be cut by two diameters, EK, GH; the rectangles of the parts of the line, are as the segments of the diameters.

For EK is as the rectangle CK . KL,
and GH is as the rectangle CH . HL;
therefore EK : GH :: CK . KL : CH . HL.

Corol. 3. If two parallels, CL, NO, be cut by two diameters, EM, GI; the rectangles of the parts of the parallels, will be as the segments of the respective diameters.

For - - - EK : EM :: CK . KL : NM . MO,
and - - - EK : GH :: CK . KL : CH . HL,
theref. by equal. EM : GH :: NM . MO : CH . HL.

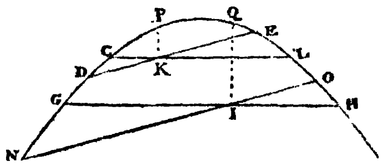
Corol. 4. When the parallels come into the position of the tangent at P, their two extremities, or points in the curve, unite in the point of contact P; and the rectangle of the parts becomes the square of the tangent, and the same properties still follow them.

So that, EV : PV :: PV : p the param.
GW : PW :: PW : p,
EV : GW :: PV^2 : PW^2,
EV : GH :: PV^2 : CH . HL.

THEOREM XXIV (20).

If two Parallels intersect any other two Parallels; the Rectangles of the Segments will be respectively Proportional.

That is, CK . KL : DK . KE :: GI . IH : NI . IO.



For, by cor. 3 theor. 23, PK : QI :: CK . KL : GI . IH;
and by the same, PK : QI :: DK . KE : NI . IO;
theref. by equal. CK . KL : DK . KE :: GI . IH : NI . IO.

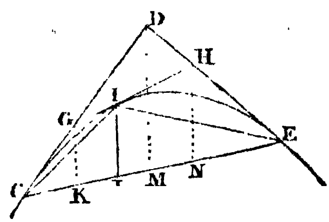
Corol. When one of the pairs of intersecting lines comes into the position of their parallel tangents, meeting and limiting each other, the rectangles of their segments become the squares of their respective tangents. So that the constant ratio of the rectangles, is that of the square of their parallel tangents, namely,

CK . KL : DK . KE :: tang^2. parallel to CL : tang^2. parallel to DE.

THEOREM XXV (21).

If there be Three Tangents intersecting each other; their Segments will be in the same Proportion.

That is, GI : IH :: CG : GD :: DH : HE.



For, through the points G, I, D, H, draw the diameters GK, IL, DM, HN; as also the lines CI, EI, which are double ordinates to the diameters GK, HN, by cor. 1 theor. 16; therefore the diameters GK, DM, HN, bisect the lines CL, CE, LE;

hence KM = CM − CK = $\frac{1}{2}$CE − $\frac{1}{2}$CL = $\frac{1}{2}$LE = LN or NE,
and MN = ME − NE = $\frac{1}{2}$CE − $\frac{1}{2}$LE = $\frac{1}{2}$CL = CK or KL,

But, by parallels, GI : IH :: KL : LN,
and - - CG : GD :: CK : KM,
also - - DH : HE :: MN : NE.

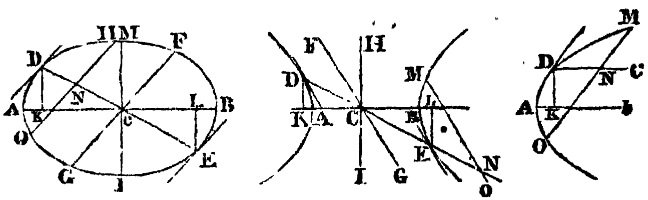
But the 3d terms KL, CK, MN are all equal;
as also the 4th terms LN, KM, NE.

Therefore the first and second terms, in all the lines, are proportional, namely GI : IH :: CG : GD :: DH : HE. Q.E.D.

SECTION

SECTION IV.

ON THE CONIC SECTIONS AS EXPRESSED BY ALGEBRAIC EQUATIONS, CALLED THE EQUATIONS OF THE CURVE.



1. *For the Ellipse.*

Let d denote AB, the transverse, or any diameter;
c = IH its conjugate;
x = AK, any absciss, from the extremity of the diam.
y = DK the correspondent ordinate.
Then, theor. 2, $AB^2 : HI^2 :: AK \cdot KB : DK^2$,
that is, $d^2 : c^2 :: x(d - x) : y^2$, hence $d^2y^2 = c^2(dx - x^2)$,
or $dy = c\sqrt{(dx - x^2)}$, the equation of the curve.

And from these equations, any one of the four letters or quantities, d, c, x, y, may easily be found, by the reduction of equations, when the other three are given.

Or, if p denote the parameter, $= c^2 \div d$ by its definition; then, by cor. th. 2, $d : p :: x(d-x) : y^2$, or $dy^2 = p(dx - x^2)$, which is another form of the equation of the curve.

Otherwise.

Or, if d = AC the semiaxis; c = CH the semiconjugate; $p = c^2 \div d$ the semiparameter; x = CK the absciss counted from the centre; and y = DK the ordinate as before.
Then is AK $= d - x$, and KB $= d + x$, and AK . KB $= (d - x) \times (d + x) = d^2 - x^2$.

Then, by th. 2, $d^2 : c^2 :: d^2 - x^2 : y^2$, and $d^2y^2 = c^2(d^2 - x^2)$, or $dy = c\sqrt{(d^2 - x^2)}$, the equation of the curve.

Or, $d : p :: d^2 - x^2 : y^2$, and $dy^2 = p(d^2 - x^2)$, another form of the equation to the curve; from which any one of the quantities may be found, when the rest are given.

2. *For the Hyperbola.*

Because the general property of the opposite hyperbolas, with respect to their abscisses and ordinates, is the same as that

that of the ellipse, therefore the process here is the very same as in the former case for the ellipse; and the equation to the curve must come out the same also, with sometimes only the change of the sign of a letter or term, from + to −, or from − to +, because here the abscisses lie beyond or without the transverse diameter, whereas they lie between or upon them in the ellipse. Thus, making the same notation for the whole diameter, conjugate, absciss, and ordinate, as at first in the ellipse; then, the one absciss AK being x, the other BK will be $d + x$, which in the ellipse was $d - x$; so the sign of x must be changed in the general property and equation, by which it becomes $d^2 : c^2 :: x(d + x) : y^2$; hence $d^2y^2 = c^2(dx + x^2)$ and $dy = c\sqrt{(dx + x^2)}$, the equation of the curve.

Or using p the parameter, as before, it is, $d : p :: x(d+x) : y^2$, or $dy^2 = p(dx + x^2)$, another form of the equation to the curve.

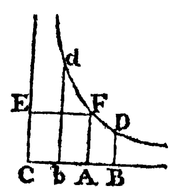
Otherwise, by using the same letters d, c, p, for the halves of the diameters and parameter, and x for the absciss CK counted from the centre; then is AK$= x - d$, and BK$= x + d$, and the property $d^2 : c^2 :: (x - d) \times (x + d) : y^2$, gives $d^2y^2 = c^2(x^2 - d^2)$, or $dy = c\sqrt{(x^2 - d^2)}$, where the signs of d^2 and x^2 are changed from what they were in the ellipse.

Or again, using the semiparameter, $d : p :: x^2 - d^2 : y^2$, and $dy^2 = p(x^2 - d^2)$ the equation of the curve.

But for the conjugate hyperbola, as in the figure to theorem 3, the signs of both x^2 and d^2 will be positive; for the property in that theorem being CA^2 : ca^2 :: CD^2 + CA^2 : De^2, it is $d^2 : c^2 :: x^2 + d^2 : y^2 =$ De^2, or $d^2y^2 = c^2(x^2 + d^2)$, and $dy = c\sqrt{(x^2 + d^2)}$, the equation to the conjugate hyperbola.

Or, as $d : p :: x^2 + d^2 : y^2$, and $dy^2 = p(x^2 + d^2)$ also the equation to the same curve.

On the Equation to the Hyperbola between the Asymptotes.



Let CE and CB be the two asymptotes to the hyperbola dFD, its vertex being F, and EF, bd, AF, BD ordinates parallel to the asymptotes. Put AF or EF $= a$, CB $= x$, and BD $= y$. Then, by theor. 28, AF . EF = CB . BD, or $a^2 = xy$, the equation to the hyperbola, when the abscisses and ordinates are taken parallel to the asymptotes.

3. *For the Parabola.*

If x denote any absciss beginning at the vertex, and y its ordinate, also p the parameter. Then, by cor. theorem 1,

AK

AK : KD : : KD : p, or $x : y :: y : p$; hence $px = y^2$ is the equation to the parabola.

4. *For the Circle.*

Because the circle is only a species of the ellipse, in which the two axes are equal to each other; therefore, making the two diameters d and c equal, in the foregoing equations to the ellipse, they become $y^2 = dx - x^2$, when the absciss x begins at the vertex of the diameter: and $y^2 = d^2 - x^2$, when the absciss begins at the centre.

Scholium.

In every one of these equations, we perceive that they rise to the 2d or quadratic degree, or to two dimensions; which is also the number of points in which every one of these curves may be cut by a right line. Hence it is also that these four curves are said to be lines of the 2d order. And these four are all the lines that are of that order, every other curve being of some higher, or having some higher equation, or may be cut in more points by a right line.

CHAPTER II.

ELEMENTS OF ISOPERIMETRY.

Def. 1. When a variable quantity has its mutations regulated by a certain law, or confined within certain limits, it is called a *maximum* when it has reached the greatest magnitude it can possibly attain; and, on the contrary, when it has arrived at the least possible magnitude, it is called a *minimum*.

Def. 2. *Isoperimeters*, or *Isoperimetrical Figures*, are those which have equal perimeters.

Def. 3. The *Locus* of any point, or intersection, &c, is the right line or curve in which these are always situated.

The problem in which it is required to find, among figures of the same or of different kinds, those which, within equal perimeters, shall comprehend the greatest surfaces, has long engaged the attention of mathematicians. Since the admirable invention of the method of Fluxions, this problem has been elegantly treated by some of the writers on that branch of

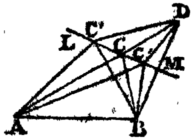
of analysis; especially by Maclaurin and Simpson. A much more extensive problem was investigated at the time of "the war of problems," between the two brothers John and James Bernoulli: namely, "To find, among all the isoperimetrical curves between given limits, such a curve, that, constructing a second curve, the ordinates of which shall be functions of the ordinates or arcs of the former, the area of the second curve shall be a maximum or a minimum." While, however, the attention of mathematicians was drawn to the most abstruse inquiries connected with isoperimetry, the *elements* of the subject were lost sight of. Simpson was the first who called them back to this interesting branch of research, by giving in his neat little book of Geometry a chapter on the maxima and minima of geometrical quantities, and some of the simplest problems concerning isoperimeters. The next who treated this subject in an elementary manner was Simon Lhuillier, of Geneva, who, in 1782, published his treatise *De Relatione mutua Capacitatis et Terminorum Figurarum*, &c. His principal object in the composition of that work was to supply the deficiency in this respect which he found in most of the Elementary Courses; and to determine, with regard to both the most usual surfaces and solids, those which possessed the minimum of contour with the same capacity; and, reciprocally, the maximum of capacity with the same boundary. M. Legendre has also considered the same subject, in a manner somewhat different from either Simpson or Lhuillier, in his *Elements de Géométrie*. An elegant geometrical tract, on the same subject, was also given, by Dr. Horsley, in the Philos. Trans. vol. 75, for 1775; contained also in the New Abridgment, vol. 13, page 653. The chief propositions deduced by these four geometers, together with a few additional propositions, are reduced into one system in the following theorems.

SECTION I. SURFACES.

THEOREM I.

Of all Triangles of the same Base, and whose Vertices fall in a right Line given in Position, the one whose Perimeter is a Minimum is that whose sides are equally inclined to that Line.

Let AB be the common base of a series of triangles ABC′, ABC, &c, whose vertices C′, C, fall in the right line LM, given in



in position, then is the triangle of least perimeter that whose sides AC, BC, are inclined to the line LM in equal angles.

For, let BM be drawn from B, perpendicularly to LM, and produced till DM = BM : join AD, and from the point C where AD cuts LM draw BC : also, from any other point C′, assumed in LM, draw C′M, C′B, C′D. Then the triangles DMC, BMC, having the angle DCM = angle ACL (th. 7 Geom.) = MCB (by hyp.), DMC = BMC, and DM = BM, and MC common to both, have also DC = BC (th. 1 Geom.).

So also, we have C′D = C′B. Hence AC + CB = AC + CD = AD, is less than AC′ + C′D (theor. 10 Geom.), or than its equal AC′ + C′B. And consequently, AB + BC + AC is less than AB + BC′ + AC′. Q. E. D.

Cor. 1. Of all triangles of the same base and the same altitude, or of all equal triangles of the same base, the isosceles triangle has the smallest perimeter.

For, the locus of the vertices of all triangles of the same altitude will be a right line LM *parallel* to the base; and when LM in the above figure becomes parallel to AB, since MCB = ACL, MCB = CBA (th. 12 Geom.), ACL = CAB; it follows that CAB = CBA, and consequently AC = CB (th. 4 Geom.).

Cor. 2. Of all triangles of the same surface, that which has the minimum perimeter is equilateral.

For the triangle of the smallest perimeter, with the same surface, must be isosceles, whichever of the sides be considered as base: therefore, the triangle of smallest perimeter has each two or each pair of its sides equal, and consequently it is equilateral.

Cor. 3. Of all rectilinear figures, with a given magnitude and a given number of sides, that which has the smallest perimeter is equilateral.

For so long as any two adjacent sides are not equal, we may draw a diagonal to become a base to those two sides, and then draw an isosceles triangle equal to the triangle so cut off, but of less perimeter: whence the corollary is manifest.

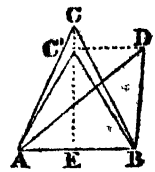
Scholium.

To illustrate the second corollary above, the student may proceed thus: assuming an isosceles triangle whose base is *not* equal to either of the two sides, and then, taking for a new base one of those sides of that triangle, he may construct another isosceles triangle equal to it, but of a smaller perimeter. Afterwards, if the base and sides of this second isosceles tri-

angle are not respectively equal, he may construct a third isosceles triangle equal to it, but of a still smaller perimeter: and so on. In performing these successive operations, he will find that the new triangles will approach nearer and nearer to an equilateral triangle.

THEOREM II.

Of all Triangles of the Same Base, and of Equal Perimeters, the Isosceles Triangle has the Greatest Surface.



Let ABC, ABD, be two triangles of the same base AB and with equal perimeters, of which the one ABC is isosceles, the other is not: then the triangle ABC has a surface (or an altitude) greater than the surface (or than the altitude) of the triangle ABD.

Draw C′D through D, parallel to AB, to cut CE (drawn perpendicular to AB) in C′: then it is to be demonstrated that CE is greater than C′E.

The triangles AC′B, ADB, are equal both in base and altitude; but the triangle AC′B is isosceles, while ADB is scalene: therefore the triangle AC′B has a smaller perimeter than the triangle ADB (th. 1 cor. 1), or than ACB (by hyp.). Consequently AC′ < AC; and in the right-angled triangles AEC′, AEC, having AE common, we have C′E < CE*. Q. E. D.

Cor. Of all isoperimetrical figures, of which the number of sides is given, that which is the greatest has all its sides equal. And in particular, of all isoperimetrical triangles, that whose surface is a maximum, is equilateral.

For, so long as any two adjacent sides are not equal, the surface may be augmented without increasing the perimeter.

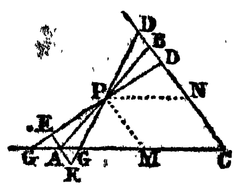
Remark. Nearly as in this theorem may it be proved that, of all triangles of equal heights, and of which the sum of the two sides is equal, that which is isosceles has the greatest base. And, of all triangles standing on the same base and having equal vertical angles, the isosceles one is the greatest.

* When two mathematical quantities are separated by the character <, it denotes that the preceding quantity is *less than* the succeeding one: when, on the contrary, the separating character is >, it denotes that the preceding quantity is *greater than* the succeeding one.

THEOREM

THEOREM III.

Of all Right Lines that can be drawn through a Given Point, between Two Right Lines Given in Position, that which is Bisected by the Given Point forms with the other two Lines the Least Triangle.



Of all right lines GD, AB, GD, that can be drawn through a given point P to cut the right lines CA, CD, given in position, that, AB, which is bisected by the given point P, forms with CA, CD, the least triangle, ABC.

For, let EE be drawn through A parallel to CD, meeting DG (produced if necessary) in E; then the triangles PBD, PAE, are manifestly equiangular; and, since the corresponding sides PB, PA are equal, the triangles are equal also. Hence PBD will be less or greater than PAG, according as CG is greater or less than CA. In the former case, let PACD, which is common, be added to both; then will BAC be less than DGC (ax. 4 Geom.). In the latter case, if PGCB be added, DCG will be greater than BAC; and consequently in this case also BAC is less than DCG. Q. E. D.

Cor. If PM and PN be drawn parallel to CB and CA respectively, the two triangles PAM, PBN, will be equal, and these two taken together (since AM = PN = MC) will be equal to the parallelogram PMCN: and consequently the parallelogram PMCN is equal to half ABC, but less than half DGC. From which it follows (consistently with both the algebraical and geometrical solution of prob. 8, Application of Algebra to Geometry), that a parallelogram is always less than half a triangle in which it is inscribed, except when the base of the one is half the base of the other, or the height of the former half the height of the latter; in which case the parallelogram is just half the triangle: this being the maximum parallelogram inscribed in the triangle.

Scholium.

From the preceding corollary it might easily be shown, that the least triangle which can possibly be described about, and the greatest parallelogram which can be inscribed in, any curve concave to its axis, will be when the subtangent is equal to half the base of the triangle, or to the whole base of the parallelogram: and that the two figures will be in the ratio of 2 to 1. But this is foreign to the present enquiry.

THEOREM IV.

Of all Triangles in which two Sides are Given in Magnitude, the Greatest is that in which the two Given Sides are Perpendicular to each other.

For, assuming for base one of the given sides, the surface is proportional to the perpendicular let fall upon that side from the opposite extremity of the other given side: therefore, the surface is the greatest when that perpendicular is the greatest; that is to say, when the other side is not inclined to that perpendicular, but *coincides* with it: hence the surface is a maximum when the two given sides are perpendicular to each other.

Otherwise. Since the surface of a triangle, in which two sides are given, is proportional to the sine of the angle included between those two sides; it follows, that the triangle is the greatest when that sine is the greatest: but the greatest sine is the sine total, or the sine of a quadrant; therefore the two sides given make a quadrantal angle, or are perpendicular to each other. Q. E. D.

THEOREM V.

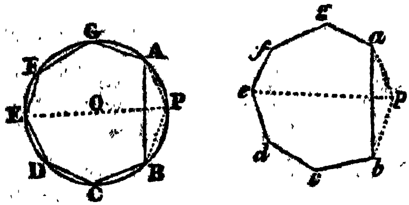
Of all Rectilinear Figures in which all the Sides except one are known, the Greatest is that which may be Inscribed in a Semicircle whose Diameter is that Unknown Side.

For, if you suppose the contrary to be the case, then whenever the figure made with the sides given, and the side unknown, is not inscribable in a semicircle of which this latter is the diameter, viz. whenever any one of the angles, formed by lines drawn from the extremities of the unknown side to one of the summits of the figure, is not a right angle; we may make a figure greater than it, in which that angle shall be right, and which shall only differ from it in that respect: therefore, whenever all the angles, formed by right lines drawn from the several vertices of the figure to the extremities of the unknown line, are not right angles, or do not fall in the circumference of a semicircle, the figure is not in its maximum state. Q. E. D.

THEOREM VI.

Of all Figures made with Sides Given in Number and Magnitude, that which may be Inscribed in a Circle is the Greatest.

Let



Let ABCDEFG be the polygon inscribed, and *abcdefg* a polygon with equal sides, but not inscribable in a circle; so that AB $=$ *ab*, BC $=$ *bc*, &c; it is affirmed that the polygon ABCDEFG is greater than the polygon *abcdefg*.

Draw the diameter EP; join AP, PB; upon *ab* $=$ AB make the triangle *abp*, equal in all respects to ABP; and join *ep*. Then, of the two figures *edcbp*, *pagfe*, one at least is not (by hyp.) inscribable in the semicircle of which *ep* is the diameter. Consequently, one at least of these two figures is smaller than the corresponding part of the figure APBCDEFG (th. 5). Therefore the figure APBCDEFG is greater than the figure *apbcdefg*: and if from these there be taken away the respective triangles APB, *apb*, which are equal by construction, there will remain (ax. 5 Geom.) the polygon ABCDEFG greater than the polygon *abcdefg*. Q. E. D.

THEOREM VII.

The Magnitude of the Greatest Polygon which can be contained under any Number of Unequal Sides, does not at all depend on the Order in which those Lines are connected with each other.

For, since the polygon is a maximum under given sides, it is inscribable in a circle (th. 6). And this inscribed polygon is constituted of as many isosceles triangles as it has sides, those sides forming the bases of the respective triangles, the other sides of all the triangles being radii of the circle, and their common summit the centre of the circle. Consequently, the magnitude of the polygon, that is, of the assemblage of these triangles, does not at all depend on their disposition, or arrangement around the common centre. Q. E. D.

THEOREM VIII.

If a Polygon Inscribed in a Circle have all its Sides Equal, all its Angles are likewise Equal, or it is a Regular Polygon.

For, if lines be drawn from the several angles of the polygon, to the centre of the circumscribing circle, they will divide the polygon into as many isosceles triangles as it has sides; and each of these isosceles triangles will be equal to either of the others in all respects, and of course they will have

have the angles at their bases all equal: consequently, the angles of the polygon, which are each made up of two angles at the bases of two contiguous isosceles triangles, will be equal to one another. Q. E. D.

THEOREM IX.

Of all Figures having the Same Number of Sides and Equal Perimeters, the Greatest is Regular.

For, the greatest figure under the given conditions has all its sides equal (th. 2 cor.). But since the sum of the sides and the number of them are given, each of them is given: therefore (th. 6), the figure is inscribable in a circle: and consequently (th. 8) all its angles are equal; that is, it is regular. Q. E. D.

Cor. Hence we see that regular polygons possess the property of a maximum of surface, when compared with any other figures of the same name and with equal perimeters.

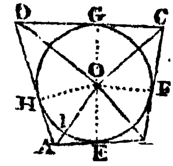
THEOREM X.

A Regular Polygon has a Smaller Perimeter than an Irregular one Equal to it in Surface, and having the Same Number of Sides.

This is the converse of the preceding theorem, and may be demonstrated thus: Let R and I be two figures equal in surface and having the same number of sides, of which R is regular, I irregular: let also R′ be a regular figure similar to R, and having a perimeter equal to that of I. Then (th. 9) $R' > I$; but $I = R$; therefore $R' > R$. But R′ and R are similar; consequently, perimeter of $R' >$ perimeter of R; while per. $R' =$ per. I (by hyp.). Hence, per. $I >$ per R. Q. E. D.

THEOREM XI.

The Surfaces of Polygons, Circumscribed about the Same or Equal Circles, are respectively as their Perimeters *.



Let the polygon ABCD be circumscribed about the circle EFGH; and let this polygon be divided into triangles, by lines drawn from its several angles to the centre O of the circle. Then, since each of the tangents AB, BC, &c, is perpendicular to its

* This theorem, together with the analogous ones respecting bodies circumscribing cylinders and spheres, were given by Emerson in his Geometry, and their use in the theory of Isoperimeters was just suggested: but the full application of them to that theory is due to Simon Lhuillier.

corre-

corresponding radius OE, OF, &c, drawn to the point of contact (th. 46 Geom.); and since the area of a triangle is equal to the rectangle of the perpendicular and half the base (Mens. of Surfaces, pr. 2); it follows, that the area of each of the triangles ABO, BCO, &c, is equal to the rectangle of the radius of the circle and half the corresponding side AB, BC, &c: and consequently, the area of the polygon ABCD, circumscribing the circle, will be equal to the rectangle of the radius of the circle and half the perimeter of the polygon. But, the surface of the circle is equal to the rectangle of the radius and half the circumference (th. 94 Geom.). Therefore, the surface of the circle, is to that of the polygon, as half the circumference of the former, to half the perimeter of the latter; or, as the circumference of the former, to the perimeter of the latter. Now, let P and P′ be any two polygons circumscribing a circle C: then, by the foregoing, we have

surf. C : surf. P :: circum. C : perim. P.

surf. C : surf. P′ :: circum. C : perim. P′.

But, since the antecedents of the ratios in both these proportions, are equal, the consequents are proportional: that is, surf. P : surf. P′ :: perim. P : perim. P′. Q. E. D.

Cor. 1. Any one of the triangular portions ABO, of a polygon circumscribing a circle, is to the corresponding circular sector, as the side AB of the polygon, to the arc of the circle included between AO and BO.

Cor. 2. Every circular arc is greater than its chord, and less than the sum of the two tangents drawn from its extremities and produced till they meet.

The first part of this corollary is evident, because a right line is the shortest distance between two given points. The second part follows at once from this proposition: for EA + AH being to the arch EIH, as the quadrangle AEOH to the circular sector HIEO; and the quadrangle being greater than the sector, because it contains it; it follows that EA + AH is greater than the arch EIH *.

Cor. 3. Hence also, any single tangent EA, is greater than its corresponding arc EI.

* This second corollary is introduced, not because of its immediate connection with the subject under discussion, but because, notwithstanding its simplicity, some authors have employed whole pages in attempting its demonstration, and failed at last.

THEOREM XII.

If a Circle and a Polygon, Circumscribable about another Circle, are Isoperimeters, the Surface of the Circle is a Geometrical Mean Proportional between that Polygon and a Similar Polygon (regular or irregular) Circumscribed about that Circle.

Let C be a circle, P a polygon isoperimetrical to that circle, and circumscribable about some other circle, and P′ a polygon similar to P and circumscribable about the circle C: it is affirmed that P : C : : C : P′.

For, P : P′ :: perim². P :: perim². P′ : : circum². C : perim². P′ by th. 89, Geom. and the hypothesis.
But (th. 11) P′ : C :: per. P′ : cir. C :: per². P′ : per. P′ × cir. C.
Therefore P : C :: - - - - cir². C : per. P′ × cir. C
; ; cir. C : per. P′ : : C : P′. Q. E. D.

THEOREM XIII.

If a Circle and a Polygon, Circumscribable about another Circle, are Equal in Surface, the Perimeter of that Figure is a Geometrical Mean Proportional between the Circumference of the first Circle and the Perimeter of a Similar Polygon Circumscribed about it.

Let C = P, and let P′ be circumscribed about C and similar to C: then it is affirmed that cir. C : per. P : per P : per P′.

For, cir. C : per. P′ : : C : P′ : : P : P′ : : per². P : per². P′.
Also, per. P′ : per. P - - - :: per². P′ : per. P × per. P′.
Therefore, cir. C : per. P - - : : per². P : per. P × per. P′
: : per. P ; per. P′. Q. E. D.

THEOREM XIV.

The Circle is Greater than any Rectilinear Figure of the Same Perimeter; and it has a Perimeter Smaller than any Rectilinear Figure of the same Surface.

For, in the proportion, P : C : : C : P′, (th. 12), since C < P′, therefore P < C.
And, in the propor. cir. C : per. P : : per. P : per. P′ (th. 13),
or, cir. C : per. P′ : : cir². C : per². P,
and cir. C < per. P′;
therefore, cir². C < per². P, or cir. C < per. P. Q. E. D.

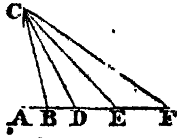
Cor. 1. It follows at once, from this and the two preceding theorems, that rectilinear figures which are isoperimeters, and

and each circumscribable about a circle, are respectively in the inverse ratio of the perimeters, or of the surfaces, of figures similar to them, and both circumscribed about one and the same circle. And that the perimeters of equal rectilineal figures, each circumscribable about a circle, are respectively in the subduplicate ratio of the perimeters, or of the surfaces, of figures similar to them, and both circumscribed about one and the same circle.

Cor. 2. Therefore, the comparison of the perimeters of equal regular figures having different numbers of sides, and that of the surfaces of regular isoperimetrical figures, is reduced to the comparison of the perimeters, or of the surfaces of regular figures respectively similar to them, and circumscribable about one and the same circle.

Lemma 1.

If an acute angle of a right-angled triangle be divided into any number of equal parts, the side of the triangle opposite to that acute angle is divided into unequal parts, which are greater as they are more remote from the right angle.



Let the acute angle C, of the right-angled triangle ACF, be divided into equal parts, by the lines CB, CD, CE, drawn from that angle to the opposite side; then shall the parts AB, BD, &c, intercepted by the lines drawn from C, be successively longer as they are more remote from the right angle A.

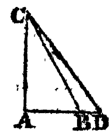
For, the angles ACD, BCE, &c, being bisected by CB, CD, &c, therefore by theor. 83 Geom. AC : CD : : AB : BD, and BC : CE : : BD : DE, and DC : CF : : DE : EF. And by th. 21 Geom. CD > CA, CE > CB, CF > CD, and so on: whence it follows, that DB > AB, DE > DB, and so on. Q. E. D.

Cor. Hence it is obvious that, if the part the most remote from the right angle A, be repeated a number of times equal to that into which the acute angle is divided, there will result a quantity greater than the side opposite to the divided angle.

THEOREM XV.

If two Regular Figures, Circumscribed about the Same Circle, differ in their Number of Sides by Unity, that which has the Greatest Number of Sides shall have the Smallest Perimeter.

Let CA be the radius of a circle, and AB, AD, the half sides of two regular polygons circumscribed about that circle, of which



which the number of sides differ by unity, being respectively $n+1$ and n. The angles ACB, ACD, therefore are respectively the $\frac{1}{n+1}$ and the $\frac{1}{n}$th part of two right angles: consequently these angles are as n and $n+1$: and hence, the angle may be conceived divided into $n+1$ equal parts, of which BCD is one. Consequently, (cor. to the lemma) $(n+1)\text{BD} > \text{AD}$. Taking, then, unequal quantities from equal quantities, we shall have

$$(n+1)\,\text{AD} - (n+1)\,\text{BD} < (n+1)\,\text{AD} - \text{AD},$$

$$\text{or, } (n+1)\,\text{AB} < n\,.\,\text{AD}.$$

That is, the semiperimeter of the polygon whose half side is AB, is smaller than the semiperimeter of the polygon whose half side is AD: whence the proposition is manifest.

Cor. Hence, augmenting successively by unity the number of sides, it follows generally, that the perimeters of polygons circumscribed about any proposed circle, become smaller as the number of their sides become greater.

THEOREM XVI.

The Surfaces of Regular Isoperimetrical Figures are Greater as the Number of their Sides is Greater: and the Perimeters of Equal Regular Figures are Smaller as the Number of their Sides is Greater.

For, 1st. Regular isoperimetrical figures are (cor. 1 th. 14) in the inverse ratio of figures similar to them circumscribed about the same circle. And (th. 15) these latter are smaller when their number of sides is greater: therefore, on the contrary, the former become greater as they have more sides.

2dly. The perimeters of equal regular figures are (cor. 1 th. 14) in the subduplicate ratio of the perimeters of similar figures circumscribed about the same circle: and (th. 15) these latter are smaller as they have more sides: therefore the perimeters of the former also are smaller when the number of their sides is greater. Q. E. D.

SECTION II. SOLIDS.

THEOREM XVII.

Of all Prisms of the Same Altitude, whose Base is Given in Magnitude and Species, or Figure, or Shape, the Right Prism has the Smallest Surface.

For,

For, the area of each face of the prism is proportional to its height; therefore the area of each face is the smallest when its height is the smallest, that is to say, when it is equal to the altitude of the prism itself: and in that case the prism is evidently a right prism. Q. E. D.

THEOREM XVIII.

Of all Prisms whose Base is Given in Magnitude and Species, and whose Lateral Surface is the Same, the Right Prism has the Greatest Altitude, or the Greatest Capacity.

This is the converse of the preceding theorem, and may readily be proved after the manner of theorem 2.

THEOREM XIX.

Of all Right Prisms of the Same Altitude, whose Bases are Given in Magnitude and of a Given Number of Sides, that whose Base is a Regular Figure has the Smallest Surface.

For, the surface of a right prism of given altitude, and base given in magnitude, is evidently proportional to the perimeter of its base. But (th. 10) the base being given in magnitude, and having a given number of sides, its perimeter is smallest when it is regular: whence, the truth of the proposition is manifest.

THEOREM XX.

Of Two Right Prisms of the Same Altitude, and with Irregular Bases Equal in Surface, that whose Base has the Greatest Number of Sides has the Smallest Surface: and, in particular, the Right Cylinder has a Smaller Surface than any Prism of the Same Altitude and the Same Capacity.

The demonstration is analogous to that of the preceding theorem, being at once deducible from theorems 16 and 14.

THEOREM XXI.

Of all Right Prisms whose Altitudes and whose Whole Surfaces are Equal, and whose Bases have a Given Number of Sides; that whose Base is a Regular Figure is the Greatest.

Let P, P′, be two right prisms of the same name, equal in altitude, and equal whole surface, the first of these having a regular, the second an irregular base; then is the base of the prism P′, less than the base of the prism P.

For, let P″ be a prism of equal altitude, and whose base is equal to that of the prism P′ and similar to that of the prism P.

Then,

Then, the lateral surface of the prism P″ is smaller than the lateral surface of the prism P′ (th. 19): hence, the total surface of P″ is smaller than the total surface of P′, and therefore (by hyp.) smaller than the whole surface of P. But the prisms P″ and P have equal altitudes and similar bases; therefore the dimensions of the base of P″ are smaller than the dimensions of the base of P. Consequently the base of P″, or that of P′, is less than the base of P; or the base of P greater than that of P′. Q. E. D.

THEOREM XXII.

Of Two Right Prisms, having Equal Altitudes, Equal Total Surfaces, and Regular Bases, that whose Base has the Greatest Number of Sides, has the Greatest Capacity. And, in particular, a Right Cylinder is Greater than any Right Prism of Equal Altitude and Equal Total Surface.

The demonstration of this is similar to that of the preceding theorem, and flows from th. 20.

THEOREM XXIII.

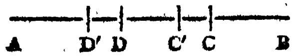
The Greatest Parallelopiped which can be contained under the Three Parts of a Given Line, any way taken, will be that constituted of Equal length, breadth, and depth.

For, let AB be the given line, and, if possible, let two parts AE, ED, be unequal. Bisect AD in C, then will the rectangle under AE ($=$ AC $+$ CE) and ED ($=$ AC $-$ CE), be less than AC^2, or than AC . CD, by the square of CE (th. 33 Geom.). Consequently, the solid AE . ED . DB, will be less than the solid AC . CD . DB; which is repugnant to the hypothesis.

Cor. Hence, of all the rectangular parallelopipeds, having the sum of their three dimensions the same, the cube is the greatest.

THEOREM XXIV.

The Greatest Parallelopiped that can possibly be contained under the Square of one Part of a Given Line, and the other Part, any way taken, will be when the former Part is the Double of the latter.



Let AB be a given line, and AC $=$ 2CB, then is AC^2 . CB the greatest possible.

For,

For, let AC' and $C'B$ be any other parts into which the given line AB may be divided; and let AC, AC', be bisected in D, D', respectively. Then shall $AC^2 . CB = 4AD . DC . CB$ (cor. to theor. 31 Geom.) $> 4AD' . D'C . CB$, or greater than its equal $C'A^2 . C'B$, by the preceding theorem.

THEOREM XXV.

Of all Right Parallelopipeds Given in Magnitude, that which has the Smallest Surface has all its Faces Squares, or is a Cube. And reciprocally, of all Parallelopipeds of Equal Surface, the Greatest is a Cube.

For, by theorems 19 and 21, the right parallelopiped having the smallest surface with the same capacity, or the greatest capacity with the same surface, has a square for its base. But, any face whatever may be taken for base: therefore, in the parallelopiped whose surface is the smallest with the same capacity, or whose capacity is the greatest with the same surface, any two opposite faces whatever are squares: consequently, this parallelopiped is a cube.

THEOREM XXVI.

The Capacities of Prisms Circumscribing the Same Right Cylinder, are Respectively as their Surfaces, whether Total or Lateral.

For, the capacities are respectively as the bases of the prisms; that is to say (th. 11), as the perimeters of their bases; and these are manifestly as the lateral surfaces: whence the proposition is evident.

Cor. The surface of a right prism circumscribing a cylinder, is to the surface of that cylinder, as the capacity of the former, to the capacity of the latter.

Def. The Archimedean cylinder is that which circumscribes a sphere, or whose altitude is equal to the diameter of its base.

THEOREM XXVII.

The Archimedean Cylinder has a Smaller Surface than any other Right Cylinder of Equal Capacity; and it is Greater than any other Right Cylinder of Equal Surface.

Let C and C' denote two right cylinders, of which the first is Archimedean, the other not: then,

1st, If . . . $C = C'$, surf. $C <$ surf. C':
2dly, if surf. $C =$ surf. C', $C > C'$.

For,

For, having circumscribed about the cylinders C, C', the right prisms P, P', with square bases, the former will be a cube, the second not: and the following series of equal ratios will obtain, viz, C : P : : surf. C : surf. P : : base C : base P : : base C' : base P' : : C' : P' : : surf. C' : surf. P'.

Then, 1st: when C = C'. Since C : P : : C' : P', it follows that P = P'; and therefore (th. 25) surf. P < surf. P'. But, surf. C : surf. P : : surf. C' : surf. P'; consequently surf. C < surf. C'. Q. E. 1D.

2dly: when surf. C = surf. C'. Then, since surf. C : surf. P : : surf. C' : surf. P', it follows that surf. P = surf. P'; and therefore (th. 25) P > P'. But C : P : : C' : P'; consequently C > C'. Q. E. 2D.

THEOREM XXVIII.

Of all Right Prisms whose Bases are Circumscribable about Circles, and Given in Species, that whose Altitude is Double the Radius of the Circle Inscribed in the Base, has the Smallest Surface with the Same Capacity, and the Greatest Capacity with the Same Surface.

This may be demonstrated exactly as the preceding theorem, by supposing cylinders inscribed in the prisms.

Scholium.

If the base cannot be circumscribed about a circle, the right prism which has the minimum surface, or the maximum capacity, is that whose lateral surface is quadruple of the surface of one end, or that whose lateral surface is two-thirds of the total surface. This is manifestly the case with the Archimedean cylinder; and the extension of the property depends solely on the mutual connexion subsisting between the properties of the cylinder, and those of circumscribing prisms.

THEOREM XXIX.

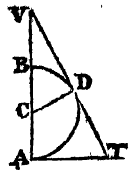
The Surfaces of Right Cones Circumscribed about a Sphere, are as their Solidities.

For, it may be demonstrated, in a manner analogous to the demonstrations of theorems 11 and 26, that these cones are equal to right cones whose altitude is equal to the radius of the inscribed sphere, and whose bases are equal to the total surfaces of the cones: therefore the surfaces and solidities are proportional.

THEOREM

THEOREM XXX.

The Surface or the Solidity of a Right Cone Circumscribed about a Sphere, is Directly as the Square of the Cone's Altitude, and Inversely as the Excess of that Altitude over the Diameter of the Sphere.



Let VAT be a right-angled triangle which, by its rotation upon VA as an axis, generates a right cone; and BDA the semicircle which by a like rotation upon VA forms the inscribed sphere: then, the surface or the solidity of the cone varies as $\frac{VA^2}{VB}$.

For, draw the radius CD to the point of contact of the semicircle and VT. Then, because the triangles VAT, VDC, are similar, it is AT : VT :: CD : VC.

And, by compos. AT : AT + VT :: CD : CD + CV = VA;

Therefore $AT^2 : (AT + VT)\,AT :: CD : VA$, by multiplying the terms of the first ratio by AT.

But, because VB, VD, VA are continued proportionals, it is $VB : VA :: VD^2 : VA^2 :: CD^2 : AT^2$ by sim. triangles.

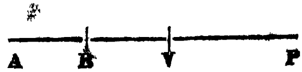
But $CD : VA :: AT^2 : (AT + VT)\,AT$ by the last; and these mult. give $CD \cdot VB : VA^2 :: CD^2 : (AT + VT)AT$,

$$\text{or } VB : CD :: VA^2 : (AT + VT)AT = CD \cdot \frac{VA^2}{VB}.$$

But the surface of the cone, which is denoted by $\pi \cdot AT^2 + \pi \cdot AT \cdot VT$ *, is manifestly proportional to the first member of this equation, is also proportional to the second member, or, since CD is constant, it is proportional to $\frac{AV^2}{BV}$, or to a third proportional to BV and AV. And, since the capacities of these circumscribing cones are as their surfaces (th. 29), the truth of the whole proposition is evident.

Lemma 2.

The difference of two right lines being given, the third proportional to the less and the greater of them is a minimum when the greater of those lines is double the other.



Let AV and BV be two right lines, whose difference AB is given, and let AP be a third proportional to BV and AV; then is AP a minimum when AV = 2BV.

---

* π being = 3·141593. See vol. ii. pa. 45.

For,

For, since $AP : AV :: AV : BV$;

By division $AP : AP - AV :: AV : AV - BV$;

That is, $AP : VP :: AV : AB$.

Hence, $VP \cdot AV = AP \cdot AB$.

But $VP \cdot AV$ is either $=$ or $<$ $\frac{1}{4}AP^2$ (cor. to th. 31 Geom. and th. 23 of this chapter).

Therefore $AP \cdot AB \overline{\overline{<}}\ \frac{1}{4}AP^2$: whence $4AB \overline{\overline{<}}\ AP$, or $AP \overline{\overline{>}}\ 4AB$. Consequently, the minimum value of AP is the quadruple of AB; and in that case $PV = VA = 2AB$. Q. E. D.*

THEOREM XXXI.

Of all Right Cones Circumscribed about the Same Sphere, the Smallest is that whose Altitude is Double the Diameter of the Sphere.

For, by th. 30, the solidity varies as $\frac{VA^2}{VB}$ (see the fig. to that theorem): and, by lemma 2, since $VA - VB$ is given, the third proportional $\frac{VA^2}{VB}$ is a minimum when $VA = 2AB$. Q. E. D.

Cor. 1. Hence, the distance from the centre of the sphere to the vertex of the least circumscribing cone, is triple the radius of the sphere.

Cor. 2. Hence also, the side of such cone is triple the radius of its base.

THEOREM XXXII.

The Whole Surface of a Right Cone being Given, the Inscribed Sphere is the Greatest when the Slant Side of the Cone is Triple the Radius of its Base.

For, let C and C' be two right cones of equal whole surface, the radii of their respective inscribed spheres being

* Though the evidence of a single demonstration, conducted on sound mathematical principles, is really irresistible, and therefore needs no corroboration, yet it is frequently conducive as well to mental improvement, as to mental delight, to obtain like results from different processes. In this view it will be advantageous to the student, to confirm the truth of several of the propositions in this chapter by means of the fluxional analysis. Let the truth enunciated in the above lemma be taken for an example: and let AB be denoted by a, AV by x, BV by $x - a$. Then we shall have $x - a : x :: x : \frac{x^2}{x-a}$, the third proportional; which is to be a minimum. Hence, the fluxion of this fraction will be equal to zero (Flux. art. 51). That is (Flux. arts. 19 and 30), $\frac{x^2\dot{x} - 2ax\dot{x}}{(x-a)^2} = 0$. Consequently $x^2 - 2ax = 0$, and $x = 2a$, or $AV = 2AB$, as above.

denoted

denoted by R and R′; let the side of the cone C be triple the radius of its base, the same ratio not obtaining in C′; and let C″ be a cone similar to C, and circumscribed about the same sphere with C′. Then, (by th. 31) surf. C″ < surf. C′; therefore surf. C″ < surf. C. But C″ and C are similar, therefore all the dimensions of C″ are less than the corresponding dimensions of C; and consequently the radius R′ of the sphere inscribed in C″ or in C′, is less than the radius R of the sphere inscribed in C, or R > R′. Q. E. D.

Cor. The capacity of a right cone being given, the inscribed sphere is the greatest when the side of the cone is triple the radius of its base.

For the capacities of such cones vary as their surfaces (th. 29).

THEOREM XXXIII.

Of all Right Cones of Equal Whole Surface, the Greatest is that whose Side is Triple the Radius of its Base: and reciprocally, of all Right Cones of Equal Capacity, that whose Side is Triple the Radius of its Base has the Least Surface.

For, by th. 29, the capacity of a right cone is in the compound ratio of its whole surface and the radius of its inscribed sphere. Therefore, the whole surface being given, the capacity is proportional to the radius of the inscribed sphere: and consequently is a maximum when the radius of the inscribed sphere is such; that is, (th. 32) when the side of the cone is triple the radius of the base*.

Again, reciprocally, the capacity being given, the surface is in the inverse ratio of the sphere inscribed; therefore, it is the smallest when that radius is the greatest; that is (th. 32) when the side of the cone is triple the radius of its base. Q. E. D.

---

* Here again a similar result may easily be deduced from the method of fluxions. Let the radius of the base be denoted by x, the slant side of the cone by z, its whole surface by a^2, and 3·141593 by π. Then the circumference of the cone's base will be $2\pi x$, its area πx^2, and the convex surface $\pi x z$. The whole surface is, therefore, $= \pi x^2 + \pi x z$; and this being $= a^2$, we have $z = \frac{a^2}{\pi x} - x$. But the altitude of the cone is equal to the square root of the difference of the squares of the side and of the radius of the base; that is, it is $= \sqrt{(\frac{a^4}{\pi^2 x^2} - \frac{2a^2}{\pi})}$. And this multiplied into $\frac{1}{3}$ of the area of the base, viz. by $\frac{1}{3}\pi x^2$, gives $\frac{1}{3}\pi x^2 \sqrt{(\frac{a^4}{\pi^2 x^2} - \frac{2a^2}{\pi})}$, for the capacity of the cone. Now,

THEOREM XXXIV.

The Surfaces, whether Total or Lateral, of Pyramids Circumscribed about the Same Right Cone, are respectively as their Solidities. And, in particular, the Surface of a Pyramid Circumscribed about a Cone, is to the Surface of that Cone, as the Solidity of the Pyramid is to the Solidity of the Cone; and these Ratios are Equal to those of the Surfaces or the Perimeters of the Bases.

For, the capacities of the several solids are respectively as their bases; and their surfaces are as the perimeters of those bases: so that the proposition may manifestly be demonstrated by a chain of reasoning exactly like that adopted in theorem 11.

THEOREM XXXV.

The Base of a Right Pyramid being Given in Species, the Capacity of that Pyramid is a Maximum with the Same Surface, and, on the contrary, the Surface is a Minimum with the Same Capacity, when the Height of One Face is Triple the Radius of the Circle Inscribed in the Base.

Let P and P′ be two right pyramids with similar bases, the height of one lateral face of P being triple the radius of the circle inscribed in the base, but this proportion not obtaining with regard to P′: then

1st. If surf. P = surf. P′, P > P′.

2dly. If . . P = . . P′, surf P < surf. P′.

For, let C and C′ be right cones inscribed within the pyramids P and P′: then, in the cone C, the slant side is triple the radius of its base, while this is not the case with respect to the cone C′. Therefore, if C = C′, surf. C < surf. C′; and, if surf. C = surf. C′, C > C′ (th. 33).

---

this being a maximum, its square must be so likewise (Flux. art. 53), that is, $\frac{a^4x^2 - 2\pi a^2x^4}{9}$, or, rejecting the denominator, as constant, $a^4x^2 - 2\pi a^2x^4$ must be a maximum. This, in fluxions, is $2a^4x\dot{x} - 8\pi a^2x^3\dot{x} = 0$; whence we have $a^2 - 4\pi x^2 = 0$, and consequently $x = \sqrt{\frac{a^2}{4\pi}}$; and $a^2 = 4\pi x^2$. Substituting this value of a^2 for it, in the value of z above given, there results $z = \frac{a^2}{\pi x} - x = \frac{4\pi x^2}{\pi x} - x = 4x - x = 3x$. Therefore, the side of the cone is triple the radius of its base. Or, the square of the altitude is to the square of the radius of the base, as 8 to 1, or, to the square of the diameter of the base, as 2 to 1.

But

But, 1st. surf. P : surf. C : : surf. P' : surf. C'; whence, if surf. P = surf. P', surf. C = surf. C'; therefore $C > C'$. But P : C : : P' : C'. Therefore $P > P'$.

2dly. $P : C :: P' : C'$. Theref. if $P = P'$, $C = C'$: consequently surf. $C <$ surf. C'. But, surf. P : surf. C : : surf. P' : surf. C'. Whence, surf. $P <$ surf. P'.

Cor. The regular tetraedron possesses the property of the minimum surface with the same capacity, and of the maximum capacity with the same surface, relatively to all right pyramids with equilateral triangular bases, and, *a fortiori*, relatively to every other triangular pyramid.

THEOREM XXXVI.

A Sphere is to any Circumscribing Solid, Bounded by Plane Surfaces, as the Surface of the Sphere to that of the Circumscribing Solid.

For, since all the planes touch the sphere, the radius drawn to each point of contact will be perpendicular to each respective plane. So that, if planes be drawn through the centre of the sphere and through all the edges of the body, the body will be divided into pyramids whose bases are the respective planes, and their common altitude the radius of the sphere. Hence, the sum of all these pyramids, or the whole circumscribing solid, is equal to a pyramid or a cone whose base is equal to the whole surface of that solid, and altitude equal to the radius of the sphere. But the capacity of the sphere is equal to that of a cone whose base is equal to the surface of the sphere, and altitude equal to its radius. Consequently, the capacity of the sphere, is to that of the circumscribing solid, as the surface of the former to the surface of the latter: both having, in this mode of considering them, a common altitude. Q. E. D.

Cor. 1. All circumscribing cylinders, cones, &c, are to the sphere they circumscribe, as their respective surfaces.

For the same proportion will subsist between their indefinitely small corresponding segments, and therefore between their wholes.

Cor. 2. All bodies circumscribing the same sphere, are respectively as their surfaces.

THEOREM XXXVII.

The Sphere is Greater than any Polyedron of Equal Surface.

For, first it may be demonstrated, by a process similar to that adopted in theorem 9, that a *regular* polyedron has a greater capacity than any other polyedron of equal surface. Let P, therefore, be a regular polyedron of equal surface to a sphere S. Then P must either circumscribe S, or fall partly within it and partly out of it, or fall entirely within it. The first of these suppositions is contrary to the hypothesis of the proposition, because in that case the surface of P could not be *equal* to that of S. Either the 2d or 3d supposition therefore must obtain; and then each plane of the surface of P must fall either partly or wholly within the sphere S: whichever of these be the case, the perpendiculars demitted from the centre of S upon the planes, will be each less than the radius of that sphere: and consequently the polyedron P must be less than the sphere S, because it has an equal base, but a less altitude. Q. E. D.

Cor. If a prism, a cylinder, a pyramid, or a cone, be equal to a sphere either in capacity, or in surface; in the first case, the surface of the sphere is less than the surface of any of those solids; in the second, the capacity of the sphere is greater than that of either of those solids.

The theorems in this chapter will suggest a variety of practical examples to exercise the student in computation. A few such are given below.

EXERCISES.

Ex. 1. Find the areas of an equilateral triangle, a square, a hexagon, a dodecagon, and a circle, the perimeter of each being 36.

Ex. 2. Find the difference between the area of a triangle whose sides are 3, 4, and 5, and of an equilateral triangle of equal perimeter.

Ex. 3. What is the area of the greatest triangle which can be constituted with two given sides 8 and 11: and what will be the length of its third side?

Ex. 4. The circumference of a circle is 12, and the perimeter of an irregular polygon which circumscribes it is 15: what are their respective areas?

Ex.

Ex. 5. Required the surface and the solidity of the greatest parallelopiped, whose length, breadth, and depth, together make 18?

Ex. 6. The surface of a square prism is 54: what is its solidity when a maximum?

Ex. 7. The content of a cylinder is 169·645968: what is its surface when a minimum?

Ex. 8. The whole surface of a right cone is 201·061952: what is its solidity when a maximum?

Ex. 9. The surface of a triangular pyramid is 43·30127: what is its capacity when a maximum?

Ex. 10. The radius of a sphere is 10. Required the solidities of this sphere, of its circumscribed equilateral cone, and of its circumscribed cylinder.

Ex. 11. The surface of a sphere is 28·274337, and of an irregular polyedron circumscribed about it 35: what are their respective solidities?

Ex. 12. The solidity of a sphere, equilateral cone, and Archimedean cylinder, are each 500: what are the surfaces and respective dimensions of each?

Ex. 13. If the surface of a sphere be represented by the number 4, the circumscribed cylinder's convex surface and whole surface will be 4 and 6, and the circumscribed equilateral cone's convex and whole surface, 6 and 9 respectively. Show how these numbers are deduced.

Ex. 14. The solidity of a sphere, circumscribed cylinder, and circumscribed equilateral cone, are as the numbers 4, 6, and 9. Required the proof.

CHAPTER III.

PLANE TRIGONOMETRY CONSIDERED ANALYTICALLY.

ART. 1. There are two methods which are adopted by mathematicians in investigating the theory of Trigonometry: the one *Geometrical*, the other *Algebraical*. In the former, the various relations of the sines, cosines, tangents, &c. of single or multiple arcs or angles, and those of the sides and angles of triangles, are deduced immediately from the figures

figures to which the several enquiries are referred; each individual case requiring its own particular method, and resting on evidence peculiar to itself. In the latter, the nature and properties of the linear-angular quantities (sines, tangents, &c,) being first defined, some general relation of these quantities, or of them in connection with a triangle, is expressed by one or more algebraical equations; and then every other theorem or precept, of use in this branch of science, is developed by the simple reduction and transformation of the primitive equation. Thus, the rules for the three fundamental cases in Plane Trigonometry, which are deduced by three independent geometrical investigations, in the second volume of this Course of Mathematics, are obtained algebraically, by forming, between the three data and the three unknown quantities, three equations, and obtaining, in expressions of known terms, the value of each of the unknown quantities, the others being exterminated by the usual processes. Each of these general methods has its peculiar advantages. The geometrical method carries conviction at every step; and by keeping the objects of enquiry constantly before the eye of the student, serves admirably to guard him against the admission of error: the algebraical method, on the contrary, requiring little aid from first principles, but merely at the commencement of its career, is more properly mechanical than mental, and requires frequent checks to prevent any deviation from truth. The geometrical method is direct, and rapid, in producing the requisite conclusions at the outset of trigonometrical science; but slow and circuitous in arriving at those results which the modern state of the science requires: while the algebraical method, though sometimes circuitous in the developement of the mere elementary theorems, is very rapid and fertile in producing those curious and interesting formulæ, which are wanted in the higher branches of pure analysis, and in mixed mathematics, especially in Physical Astronomy. This mode of developing the theory of Trigonometry is, consequently, well suited for the use of the more advanced student: and is therefore introduced here with as much brevity as is consistent with its nature and utility.

§ 2. To save the trouble of turning very frequently to the 2d volume, a few of the principal definitions, there given, are here repeated, as follows:

The SINE of an arc, is the perpendicular let fall from one of its extremities upon the diameter of the circle which passes through the other extremity.

The

The COSINE of an arc, is the sine of the complement of that arc, and is equal to the part of the radius comprised between the centre of the circle and the foot of the sine.

The TANGENT of an arc, is a line which touches the circle in one extremity of that arc, and is continued from thence till it meets a line drawn from or through the centre and through the other extremity of the arc.

The SECANT of an arc, is the radius drawn through one of the extremities of that arc and prolonged till it meets the tangent drawn from the other extremity.

The VERSED SINE of an arc, is that part of the diameter of the circle which lies between the beginning of the arc and the foot of the sine.

The COTANGENT, COSECANT, and COVERSED SINE of an arc, are the tangent, secant, and versed sine, of the complement of such arc.

3. Since arcs are proper and adequate measures of plane angles, (the ratio of any two plane angles being constantly equal to the ratio of the two arcs of any circle whose centre is the angular point, and which are intercepted by the lines whose inclinations form the angle), it is usual, and it is perfectly safe, to apply the above names without circumlocution as though they referred to the angles themselves; thus, when we speak of the sine, tangent, or secant, of an angle, we mean the sine, tangent, or secant, of the arc which measures that angle; the radius of the circle employed being known.

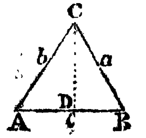
4. It has been shown in the 2d vol. (pa. 6), that the tangent is a fourth proportional to the cosine, sine, and radius; the secant, a third proportional to the cosine and radius; the cotangent, a fourth proportional to the sine, cosine, and radius; and the cosecant a third proportional to the sine and radius. Hence, making use of the obvious abbreviations, and converting the analogies into equations, we have

$$\text{tan.} = \frac{\text{rad} \times \text{sine}}{\text{cos.}}, \ \text{cot.} = \frac{\text{rad.} \times \text{cos.}}{\text{sine}}, \ \text{sec.} = \frac{\text{rad}^2}{\text{cos.}}, \ \text{cos.} = \frac{\text{rad}^2}{\text{sine.}}.$$

Or, assuming unity for the rad. of the circle, these will become

$$\text{tan.} = \frac{\text{sin.}}{\text{cos.}} \ldots \ \text{cot.} = \frac{\text{cos.}}{\text{sin.}} \ldots \ \text{sec.} = \frac{1}{\text{cos.}} \ldots \ \text{cos.} = \frac{1}{\text{sin.}}.$$

These preliminaries being borne in mind, the student may pursue his investigations.



5. Let ABC be any plane triangle, of which the side BC opposite the angle A is denoted by the small letter a, the side AC opposite the angle B by the small letter b, and the side AB opposite the angle C by

the

the small letter c, and CD perpendicular to AB: then is, $c = a \cdot \cos B + b \cdot \cos A$.

For, since AC $= b$, AD is the cosine of A to that radius; consequently, supposing radius to be unity, we have AD $= b \cdot \cos. A$. In like manner it is BD $= a \cdot \cos. B$. Therefore, AD + BD = AB $= c = a \cdot \cos. B + b \cdot \cos. A$. By pursuing similar reasoning with respect to the other two sides of the triangle, exactly analagous results will be obtained. Placed together, they will be as below:

$$\left.\begin{array}{l} a = b \cdot \cos. C + c \cdot \cos. B \\ b = a \cdot \cos. C + c \cdot \cos. A \\ c = a \cdot \cos. B + b \cdot \cos. A \end{array}\right\} \text{(I.)}$$

6. Now, if from these equations it were required to find expressions for the angles of a plane triangle, when the sides are given; we have only to multiply the first of these equations by a, the second by b, the third by c, and to subtract each of the equations thus obtained from the sum of the other two. For thus we shall have

$$\left.\begin{array}{ll} b^2 + c^2 - a^2 = 2bc \cdot \cos. A, \text{ whence } \cos. A = \frac{b^2+c^2-a^2}{2bc} \\ a^2 + c^2 - b^2 = 2ac \cdot \cos. B, \quad . \quad . \quad . \cos. B = \frac{a^2+c^2-b^2}{2ac} \\ a^2 + b^2 - c^2 = 2ab \cdot \cos. C, \quad . \quad . \quad . \cos. C = \frac{a^2+b^2-c^2}{2ab} \end{array}\right\} \text{(II.)}$$

7. More convenient expressions than these will be deduced hereafter: but even these will often be found very convenient, when the sides of triangles are expressed in integers, and tables of sines and tangents, as well as a table of squares, (like that in our first vol.) are at hand.

Suppose, for example, the sides of the triangle are $a=320$, $b = 562$, $c = 800$, being the numbers given in prop. 4, pa. 161, of the Introduction to the Mathematical Tables: then we have

$b^2 + c^2 - a^2 = 853444$ . . . . . . log. = 5·9311751
$2bc$ . . . = 899200 . . . . . . log. = 5·9538080
The remainder being log. cos. A, or of $18°20' =$ 9·9773671
Again, $a^2 + c^2 - b^2 = 426556$ . . . log. = 5·6299760
$2ac$ . . . = 512000 . . . log. = 5·7092700
The remainder being log. cos. B, or of $33°35' =$ 9.9207060
Then $180° - (18°20' + 33°35') = 128°5' =$ C; where all the three angles are determined in 7 lines.

8. If it were wished to get expressions for the sines, instead of the cosines, of the angles; it would merely be necessary to introduce into the preceding equations (marked II), instead

instead of cos. A, cos. B, &c, their equivalents cos. $\text{A} = \sqrt{(1 - \sin^2. \text{A})}$, cos. $\text{B} = \sqrt{(1 - \sin^2. \text{B})}$, &c. For then, after a little reduction, there would result,

$$\left.\begin{aligned} \sin. \text{A} &= \frac{1}{2bc}\sqrt{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4)} \\ \sin. \text{B} &= \frac{1}{2ac}\sqrt{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4)} \\ \sin. \text{C} &= \frac{1}{2ab}\sqrt{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4)} \end{aligned}\right\}$$

Or, resolving the expression under the radical into its four constituent factors, substituting s for $a+b+c$, and reducing, the equations will become

$$\left.\begin{aligned} \sin. \text{A} &= \frac{2}{bc}\sqrt{\tfrac{1}{2}\text{s}(\tfrac{1}{2}\text{s} - a)(\tfrac{1}{2}\text{s} - b)(\tfrac{1}{2}\text{s} - c)} \\ \sin. \text{B} &= \frac{2}{ac}\sqrt{\tfrac{1}{2}\text{s}(\tfrac{1}{2}\text{s} - a)(\tfrac{1}{2}\text{s} - b)(\tfrac{1}{2}\text{s} - c)} \\ \sin. \text{C} &= \frac{2}{ab}\sqrt{\tfrac{1}{2}\text{s}(\tfrac{1}{2}\text{s} - a)(\tfrac{1}{2}\text{s} - b)(\tfrac{1}{2}\text{s} - c)} \end{aligned}\right\} \quad \text{(III.)}$$

These equations are moderately well suited for computation in their latter form; they are also perfectly symmetrical: and as indeed the quantities under the radical are identical, and are constituted of known terms, they may be represented by the same character; suppose K: then shall we have

$$\sin. \text{A} = \frac{2\text{K}}{bc} \ldots \sin. \text{B} = \frac{2\text{K}}{ac} \ldots \sin. \text{C} = \frac{2\text{K}}{ab} \ldots (iii.)$$

Hence we may immediately deduce a very important theorem: for, the first of these equations, divided by the second, gives $\frac{\sin. \text{A}}{\sin. \text{B}} = \frac{a}{b}$, and the first divided by the third gives $\frac{\sin. \text{A}}{\sin. \text{C}} = \frac{a}{c}$: whence we have

$$\sin. \text{A} : \sin. \text{B} : \sin. \text{C} \propto a : b : c \ldots \text{(IV.)}$$

Or, in words, *the sides of plane triangles are proportional to the sines of their opposite angles.* (See th. 1 Trig. vol. ii).

9. Before the remainder of the theorems, necessary in the solution of plane triangles, are investigated, the fundamental proposition in the theory of sines, &c, must be deduced, and the method explained by which Tables of these quantities, confined within the limits of the quadrant, are made to extend to the whole circle, or to any number of quadrants whatever. In order to this, expressions must be first obtained for the sines, cosines, &c, of the sums and differences of any two arcs or angles. Now, it has been found (I) that $a = b \,.\, \cos. \text{C} + c \,.\, \cos. \text{B}$. And the equations (IV) give $b = a \,.\, \frac{\sin. \text{B}}{\sin. \text{A}}$, $c = a \,.\, \frac{\sin. \text{C}}{\sin. \text{A}}$. Substituting these values

lues of b and c for them in the preceding equation, and multiplying the whole by $\frac{\sin. A}{a}$, it will become

$$\sin. A = \sin. B . \cos. C + \sin. C . \cos. B.$$

But, in every plane triangle, the sum of the three angles is equal to two right angles; therefore, B and C are equal to the supplement of A: and, consequently, since an angle and its supplement have the same sine (cor. 1, pa. 3, vol. ii), we have $\sin. (B + C) = \sin. B . \cos. C + \sin. C . \cos. B.$

10. If, in the last equation, C become subtractive, then would sin. C manifestly become subtractive also, while the cosine of C would not change its sign, since it would still continue to be estimated on the same radius in the same direction. Hence the preceding equation would become

$$\sin. (B - C) = \sin. B . \cos. C - \sin. C . \cos. B.$$

11. Let C′ be the complement of C, and $\frac{1}{4}\bigcirc$ be the quarter of the circumference: then will $C' = \frac{1}{4}\bigcirc - C$, $\sin. C' = \cos. C$, and $\cos. C' = \sin. C$. But (art. 10), $\sin. (B - C') = \sin. B . \cos. C' - \sin. C' \cos. B$. Therefore, substituting for sin. C′, cos. C′, their values, there will result $\sin. (B - C') = \sin. B . \sin. C - \cos. B . \cos. C$. But because $C' = \frac{1}{4}\bigcirc - C$, we have $\sin. (B - C') = \sin. (B + C - \frac{1}{4}\bigcirc) = \sin. [(B + C) - \frac{1}{4}\bigcirc] = -\sin. [\frac{1}{4}\bigcirc - (B + C)] = -\cos. (B + C)$. Substituting this value of $\sin. (B - C')$ in the equation above, it becomes $\cos. (B + C) = \cos. B . \cos. C - \sin. B . \sin. C$.

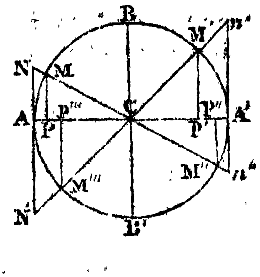
12. In this latter equation, if C be made subtractive, sin. C will become − sin. C, while cos. C will not change: consequently the equation will be transformed to the following, viz, $\cos. (B - C) = \cos. B . \cos. C + \sin. B . \sin. C$.

If, instead of the angles B and C, the angles had been A and B; or, if A and B represented the *arcs* which measure those angles, the results would evidently be similar: they may therefore be expressed generally by the two following equations, for the sines and cosines of the sums or differences of any two arcs or angles:

$$\left.\begin{aligned} \sin. (A \pm B) &= \sin. A . \cos. B \pm \sin. B . \cos. A. \\ \cos. (A \pm B) &= \cos. A . \cos. B \mp \sin. A . \sin. B. \end{aligned}\right\} \quad (V.)$$

13. We are now in a state to trace completely the mutations of the sines, cosines, &c, as they relate to arcs in the various parts of a circle; and thence to perceive that tables which apparently are included within a quadrant, are, in fact, applicable to the whole circle.

Imagine that the radius MC of the circle, in the marginal figure, coinciding at first with AC, turns about the point C (in the same manner as a rod would turn on a pivot) and thus forming



forming successively with AC all possible angles: the point M at its extremity passing over all the points of the circumference ABA'B'A, or describing the whole circle. Tracing this motion attentively, it will appear, that at the point A, where the arc is nothing, the sine is nothing also, while the cosine does not differ from the radius. As the radius MC recedes from AC, the sine PM keeps increasing, and the cosine CP decreasing, till the describing point M has passed over a quadrant, and arrived at B: in that case, PM becomes equal to CB the radius, and the cosine CP vanishes. The point M continuing its motion beyond B, the sine P'M' will diminish, while the cosine CP', which now falls on the *contrary* side of the centre C will increase. In the figure, P'M' and CP' are respectively the sine and cosine of the arc A'M', or the sine and cosine of ABM', which is the supplement of A'M' to $\frac{1}{2}$○, half the circumference: whence it follows that an obtuse angle (measured by an arc greater than a quadrant) has *the same sine and cosine as its supplement;* the cosine, however, being reckoned subtractive or negative, because it is situated contrariwise with regard to the centre C.

When the describing point M has passed over $\frac{1}{2}$○, or half the circumference, and has arrived at A', the sine P'M' vanishes, or becomes nothing, as at the point A, and the cosine is again equal to the radius of the circle. Here the angle ACM has attained its maximum limit; but the radius CM may still be supposed to continue its motion, and pass *below* the diameter AA'. The sine, which will then be P''M'', will consequently fall below the diameter, and will augment as M moves along the third quadrant, while on the contrary CP'', the cosine, will diminish. In this quadrant too, both sine and cosine must be considered as negative; the former being on a contrary side of the diameter, the latter a contrary side of the centre, to what each was respectively in the first quadrant. At the point B', where the arc is three-fourths of the circumference, or $\frac{3}{4}$○, the sine P''M'' becomes equal to the radius CB, and the cosine CP'' vanishes. Finally, in the fourth quadrant, from B' to A, the sine P'''M''', always *below* AA', diminishes in its progress, while the cosine CP''', which is then found on the same side of the centre as it was in the first quadrant, augments till it becomes equal to the radius CA. Hence, the sine in this quadrant is to be considered as negative

tive or subtractive, the cosine as positive. If the motion of M were continued through the circumference again, the circumstances would be exactly the same in the fifth quadrant as in the first, in the sixth as in the second, in the seventh as in the third, in the eighth as in the fourth: and the like would be the case in any subsequent revolutions.

14. If the mutations of the *tangent* be traced in like manner, it will be seen that its magnitude passes from nothing to infinity in the first quadrant; becomes negative, and decreases from infinity to nothing in the second; becomes positive again, and increases from nothing to infinity in the third quadrant; and lastly, becomes negative again, and decreases from infinity to nothing, in the fourth quadrant.

15. These conclusions admit of a ready confirmation, and others may be deduced, by means of the analytical expressions in arts. 4 and 12. Thus, if A be supposed equal to $\frac{1}{4}\bigcirc$, in equa. V, it will become

$$\cos.(\tfrac{1}{4}\bigcirc \pm B) = \cos.\tfrac{1}{4}\bigcirc . \cos. B \mp \sin.\tfrac{1}{4}\bigcirc . \sin. B,$$
$$\sin.(\tfrac{1}{4}\bigcirc \pm B) = \sin.\tfrac{1}{4}\bigcirc . \cos. B \pm \sin. B . \cos.\tfrac{1}{4}\bigcirc.$$

But $\sin.\frac{1}{4}\bigcirc = \text{rad.} = 1$; and $\cos.\frac{1}{4}\bigcirc = 0$: so that the above equations will become

$$\cos.(\tfrac{1}{4}\bigcirc \pm B) = \mp \sin. B.$$
$$\sin.(\tfrac{1}{4}\bigcirc \pm B) = \cos. B.$$

From which it is obvious, that if the sine and cosine of an arc, less than a quadrant, be regarded as positive, the cosine of an arc greater than $\frac{1}{4}\bigcirc$ and less than $\frac{1}{2}\bigcirc$ will be negative, but its sine positive. If B also be made $= \frac{1}{4}\bigcirc$; then shall we have $\cos.\frac{1}{2}\bigcirc = -1$; $\sin.\frac{1}{2}\bigcirc = 0$.

Suppose next, that in the equa. V, $A = \frac{1}{2}\bigcirc$; then shall we obtain

$$\cos.(\tfrac{1}{2}\bigcirc \pm B) = -\cos. B.$$
$$\sin.(\tfrac{1}{2}\bigcirc \pm B) = \mp \sin. B;$$

which indicates, that every arc comprised between $\frac{1}{2}\bigcirc$ and $\frac{3}{4}\bigcirc$, or that terminates in the third quadrant, will have its sine and its cosine both negative. In this case too, when $B = \frac{1}{4}\bigcirc$, or the arc terminates at the *end* of the third quadrant, we shall have $\cos.\frac{3}{4}\bigcirc = 0$, $\sin.\frac{3}{4}\bigcirc = -1$.

Lastly, the case remains to be considered in which $A = \frac{3}{4}\bigcirc$, or in which the arc terminates in the fourth quadrant. Here the primitive equations (V) give

$$\cos.(\tfrac{3}{4}\bigcirc \pm B) = \pm \sin. B.$$
$$\sin.(\tfrac{3}{4}\bigcirc \pm B) = -\cos. B;$$

so that in all arcs between $\frac{3}{4}\bigcirc$ and $\bigcirc$, the cosines are positive and the sines negative.

16. The changes of the tangents, with regard to positive and negative, may be traced by the application of the preceding

ceding results to the algebraic expression for the tangent; viz, tan. $= \frac{\text{sin.}}{\text{cos.}}$. For it is hence manifest, that when the sine and cosine are either both positive or both negative, the tangent will be positive; which will be the case in the first and third quadrants. But when the sine and cosine have different signs, the tangents will be negative, as in the second and fourth quadrants. The algebraic expression for the cotangent, viz, cot. $= \frac{\text{cos.}}{\text{sin.}}$, will produce exactly the same results.

The expressions for the secants and cosecants, viz, sec. $= \frac{1}{\text{cos}}$, cosec. $= \frac{1}{\text{sin.}}$ show, that the signs of the secants are the same as those of the cosines; and those of the cosecants the same as those of the sines.

The *magnitude* of the tangent at the end of the first and third quadrants will be infinite; because in those places the sine is equal to radius, the cosine equal to zero, and therefore $\frac{\text{sin.}}{\text{cos.}} = \infty$ (infinity). Of these, however, the former will be reckoned positive, the latter negative.

17. The magnitudes of the cotangents, secants, and cosecants, may be traced in like manner; and the results of the 13th, 14th, and 15th articles, recapitulated and tabulated as below.

	0°	90°	180°	270°	360°	
Sin.	0	R	0	− R	0	(VI.)
Tan.	0	∞	0	− ∞	0	
Sec.	R	∞	− R	− ∞	R	
Cos.	R	0	− R	0	R	
Cot.	∞	0	− ∞	0	∞	
Cosec.	∞	R	− ∞	− R	∞	

The changes of signs are these:

quadrants				sin.	cos.	tan.	cot.	sec.	cosec.	
1st.	5th.	9th.	13th.	+	+	+	+	+	+	(VII.)
2d.	6th.	10th.	14th.	+	−	−	−	−	+	
3d.	7th.	11th.	15th.	−	−	+	+	−	−	
4th.	8th.	12th.	16th.	−	+	−	−	+	−	

We have been thus particular in tracing the mutations, both with regard to value and algebraic signs, of the principal trigonometrical quantities, because a knowledge of them is absolutely necessary in the application of trigonometry to the solution of equations, and to various astronomical and physical problems.

18. We may now proceed to the investigation of other expressions relating to the sums, differences, multiples, &c, of

arcs:

arcs; and in order that these expressions may have the more generality, give to the radius any value R, instead of confining it to unity. This indeed may always be done in an expression, however complex, by merely rendering all the terms homogeneous; that is, *by multiplying each term by such a power of* R *as shall make it of the same dimension, as the term in the equation which has the highest dimension.* Thus, the expression for a triple arc

$$\sin.\ 3A = 3\sin.\ A - 4\sin^3.\ A \text{ (radius } = 1)$$

becomes when radius is assumed $= R$,

$$R^2 \sin.\ 3A = R^2\ 3\sin.\ A - 4\sin^3.\ A$$

$$\text{or } \sin.\ 3A = \frac{3R^2 \sin.\ A - 4\sin^3.\ A}{R^2}.$$

Hence then, if consistently with this precept, R be placed for a denominator of the second member of each equation V (art. 12), and if A be supposed equal to B, we shall have

$$\sin.\ (A + A) = \frac{\sin.\ A\ .\ \cos.\ A + \sin.\ A\ .\ \cos.\ A}{R}.$$

$$\text{That is, } \sin.\ 2A = \frac{2\sin.\ A\ .\ \cos.\ A}{R}.$$

And, in like manner, by supposing B to become successively equal to 2A, 3A, 4A, &c, there will arise

$$\left.\begin{aligned} \sin.\ 3A &= \frac{\sin.\ A\ .\ \cos.\ 2A + \cos.\ A\ .\ \sin.\ 2A}{R} \\ \sin.\ 4A &= \frac{\sin.\ A\ .\ \cos.\ 3A + \cos.\ A\ .\ \sin.\ 3A}{R} \\ \sin.\ 5A &= \frac{\sin.\ A\ .\ \cos.\ 4A + \cos.\ A\ .\ \sin.\ 4A}{R} \end{aligned}\right\} \text{(VIII.)}$$

And, by similar processes, the second of the equations just referred to, namely, that for cos. (A + B), will give successively,

$$\left.\begin{aligned} \cos.\ 2A &= \frac{\cos^2 A - \sin^2.\ A}{R} \\ \cos.\ 3A &= \frac{\cos.\ A\ .\ \cos.\ 2A - \sin.\ A\ .\ \sin.\ 2A}{R} \\ \cos.\ 4A &= \frac{\cos.\ A\ .\ \cos.\ 3A - \sin.\ A\ .\ \sin.\ 3A}{R} \\ \cos.\ 5A &= \frac{\cos.\ A\ .\ \cos.\ 4A - \sin.\ A\ .\ \sin.\ 4A}{R} \end{aligned}\right\} \text{(IX.)}$$

19. If, in the expressions for the successive multiples of the sines, the values of the several cosines in terms of the sines were substituted for them; and a like process were adopted with regard to the multiples of the cosines, other expressions would be obtained, in which the multiple sines would be expressed in terms of the radius and sine, and the multiple cosines in terms of the radius and cosine.

As

$$\left.\begin{array}{ll}
\text{As sin. A} = & s \\
\text{sin. 2A} = & 2s\sqrt{R^2-s^2} \\
\text{sin. 3A} = & 3s-4s^3 \\
\text{sin. 4A} = & (4s-8s^3)\sqrt{R^2-s^2} \\
\text{sin. 5A} = & 5s-20s^3+16s^5 \\
\text{sin. 6A} = & (6s-32s^3+32s^5)\sqrt{R^2-s^2} \\
 & \&c.\ \&c.
\end{array}\right\}\ (\text{X.})$$

$$\left.\begin{array}{ll}
\text{Cos. A} = & c \\
\text{cos. 2A} = & 2c^2-1 \\
\text{cos. 3A} = & 4c^3-3c \\
\text{cos. 4A} = & 8c^4-8c^2+1 \\
\text{cos. 5A} = & 16c^5-20c^3+5c \\
\text{cos. 6A} = & 32c^6-48c^4+18c^2-1 \\
 & \&c.\ \&c^*.
\end{array}\right\}\ (\text{XI.})$$

Other very convenient expressions for multiple arcs may be obtained thus:

Add together the expanded expressions for sin. (B + A), sin. (B − A), that is,

add - - sin. (B + A) = sin. B . cos. A + cos. B . sin. A,

to - - sin. (B − A) = sin. B . cos. A − cos. B . sin. A;

there results sin. (B + A) + sin. (B − A) = 2 cos. A . sin. B:

whence, - sin. (B + A) = 2 cos. A . sin. B − sin. (B − A).

Thus again, by adding together the expressions for cos (B + A) and cos. (B − A), we have

cos. (B + A) + cos. (B − A) = 2 cos. A . cos. B;

whence, cos. (B + A) = 2 cos. A . cos. B − cos. (B − A).

Substituting in these expressions for the sine and cosine of B + A, the successive values A, 2A, 3A, &c, instead of B; the following series will be produced.

$$\left.\begin{array}{l}
\text{sin. 2A} = 2\text{ cos. A . sin. A} \\
\text{sin. 3A} = 2\text{ cos. A . sin. 2A} - \text{sin. A.} \\
\text{sin. 4A} = 2\text{ cos. A . sin. 3A} - \text{sin. 2A.} \\
\text{sin. } n\text{A} = 2\text{ cos. A . sin.}(n-1)\text{A} - \text{sin.}(n-2)\text{A.}
\end{array}\right\}\ (x.)$$

$$\left.\begin{array}{l}
\text{cos. 2A} = 2\text{ cos. A . cos. A} - \text{cos. } 0\ (=1). \\
\text{cos. 3A} = 2\text{ cos. A . cos. 2A} - \text{cos. A.} \\
\text{cos. 4A} = 2\text{ cos. A . cos. 3A} - \text{cos. 2A.} \\
\text{cos. } n\text{A} = 2\text{ cos. A . cos.}(n-1)\text{A} - \text{cos.}(n-2)\text{A.}
\end{array}\right\}\ (xi.)$$

Several other expressions for the sines and cosines of multiple arcs, might readily be found: but the above are the most useful and commodious.

---

* Here we have omitted the powers of R that were necessary to render all the terms homologous, merely that the expressions might be brought in upon the page; but they may easily be supplied, when needed, by the rule in art. 18.

20. From

20. From the equation $\sin 2A = \frac{2 \sin A \cdot \cos A}{R}$, it will be easy, when the sine of an arc is known, to find that of its half. For, substituting for cos A its value $\sqrt{(R^2 - \sin^2 A)}$, there will arise $\sin 2A = \frac{2 \sin A \sqrt{(R^2 - \sin^2 A)}}{R}$. This squared

gives $R^2 \sin^2 2A = 4R^2 \sin^2 A - 4 \sin^4 A$.

Here taking sin A for the unknown quantity, we have a quadratic equation, which solved after the usual manner, gives

$$\sin A = \pm \sqrt{\tfrac{1}{2}R^2 \pm \tfrac{1}{2}R\sqrt{R^2 - \sin^2 2A}}.$$

If we make $2A = A'$, then will $A = \frac{1}{2}A'$, and consequently the last equation becomes

$$\left.\begin{array}{l} \sin \frac{1}{2}A' = \pm \sqrt{\frac{1}{2}R^2 \pm \frac{1}{2}R\sqrt{R^2 - \sin^2 A'}} \\ \text{or } \sin \frac{1}{2}A' = \pm \frac{1}{2}\sqrt{2R^2 \pm 2R \cos A'}: \end{array}\right\} \text{(XII.)}$$

by putting $\cos A'$ for its value $\sqrt{R^2 - \sin^2 A'}$ multiplying the quantities under the radical by 4, and dividing the whole second number by 2. Both these expressions for the sine of half an arc or angle will be of use to us as we proceed.

21. If the values of $\sin (A + B)$ and sine $(A - B)$, given by equa. V, be added together, there will result

$$\sin (A + B) + \sin (A - B) = \frac{2 \sin A \cdot \cos B}{R}; \text{ whence,}$$

$$\sin A \cdot \cos B = \tfrac{1}{2}R \cdot \sin (A + B) + \tfrac{1}{2}R \sin (A - B) \;.\,.\; \text{(XIII.)}$$

Also, taking $\sin (A - B)$ from $\sin (A + B)$, gives

$$\sin (A + B) - \sin (A - B) = \frac{2 \sin B \cdot \cos A}{R}; \text{ whence,}$$

$$\sin B \cdot \cos A = \tfrac{1}{2}R \cdot \sin (A + B) - \tfrac{1}{2}R \cdot \sin (A - B) \;.\,.\; \text{(XIV.)}$$

When $A = B$, both equa. XIII and XIV, become

$$\cos A \cdot \sin A = \tfrac{1}{2}R \sin 2A \;.\,.\; \text{(XV.)}$$

22. In like manner, by adding together the primitive expressions for $\cos (A + B)$, $\cos (A - B)$, there will arise

$$\cos (A + B) + \cos (A - B) = \frac{2 \cos A \cdot \cos B}{R}; \text{ whence,}$$

$$\cos A \cdot \cos B = \tfrac{1}{2}R \cdot \cos (A + B) + \tfrac{1}{2}R \cdot \cos (A - B) \text{ (XVI.)}$$

And here, when $A = B$, recollecting that when the arc is nothing the cosine is equal to radius, we shall have

$$\cos^2 A = \tfrac{1}{2} R \cdot \cos 2A + \tfrac{1}{2}R^2 \;.\,.\,.\; \text{(XVII.)}$$

23. Deducting $\cos (A + B)$ from $\cos (A - B)$, there will remain

$$\cos (A - B) - \cos (A + B) = \frac{2 \sin A \cdot \sin B}{R}; \text{ whence,}$$

$$\sin A \cdot \sin B = \tfrac{1}{2}R \cdot \cos (A - B) - \tfrac{1}{2}R \cdot \cos (A + B) \text{ (XVIII.)}$$

When $A = B$, this formula becomes

$$\sin^2 A = \tfrac{1}{2}R^2 - \tfrac{1}{2}R \cdot \cos 2A \;.\,.\,.\; \text{(XIX.)}$$

24. Mul-

24. Multiplying together the expressions for $\sin(A+B)$ and $\sin(A-B)$, equation V, and reducing, there results

$$\sin(A+B) \cdot \sin(A-B) = \sin^2 A - \sin^2 B.$$

And, in like manner, multiplying together the values of $\cos(A+B)$ and $\cos(A-B)$, there is produced

$$\cos(A+B) \cdot \cos(A-B) = \cos^2 A - \cos^2 B$$

Here, since $\sin^2 A - \sin^2 B$, is equal to $(\sin A + \sin B) \times (\sin A - \sin B)$, that is, to the rectangle of the sum and difference of the sines; it follows, that the first of these equations converted into an analogy, becomes

$$\sin(A-B) : \sin A - \sin B :: \sin A + \sin B : \sin(A+B) \quad \text{(XX.)}$$

That is to say, *the sine of the difference of any two arcs or angles, is to the difference of their sines, as the sum of those sines is to the sine of their sum.*

If A and B be to each other as $n+1$ to n, then the preceding proportion will be converted into $\sin A : \sin(n+1)A - \sin nA :: \sin(n+1)A + \sin nA : \sin(2n+1)A \ldots$ (XXI.)

These two proportions are highly useful in computing a table of sines; as will be shown in the practical examples at the end of this chapter.

25. Let us suppose $A + B = A'$, and $A - B = B'$; then the half sum and the half difference of these equations will give respectively $A = \frac{1}{2}(A' + B')$, and $B = \frac{1}{2}(A' - B')$ Putting these values of A and B, in the expressions of $\sin A . \cos B$, $\sin B . \cos A$, $\cos A . \cos B$, $\sin A . \sin B$, obtained in arts. 21, 22, 23, there would arise the following formulæ:

$$\sin \tfrac{1}{2}(A' + B') \cdot \cos \tfrac{1}{2}(A' - B') = \tfrac{1}{2}R(\sin A' + \sin B'),$$
$$\sin \tfrac{1}{2}(A' - B') \cdot \cos \tfrac{1}{2}(A' + B') = \tfrac{1}{2}R(\sin A' - \sin B'),$$
$$\cos \tfrac{1}{2}(A' + B') \cdot \cos \tfrac{1}{2}(A' - B') = \tfrac{1}{2}R(\cos A' + \cos B'),$$
$$\sin \tfrac{1}{2}(A' + B') \cdot \sin \tfrac{1}{2}(A' - B') = \tfrac{1}{2}R(\cos B - \cos A).$$

Dividing the second of these formulæ by the first, there will be had

$$\frac{\sin \frac{1}{2}(A' - B') \cdot \cos \frac{1}{2}(A' + B')}{\sin \frac{1}{2}(A' + B') \cdot \cos \frac{1}{2}(A' - B')} = \frac{\sin \frac{1}{2}(A' - B')}{\cos \frac{1}{2}(A' - B')} \cdot \frac{\cos \frac{1}{2}(A' + B')}{\sin \frac{1}{2}(A' + B')} = \frac{\sin A' - \sin B'}{\sin A' + \sin B'}.$$

But since $\frac{\sin}{\cos} = \frac{\tan}{R}$, and $\frac{\cos}{\sin} = \frac{R}{\tan}$, it follows, that the two factors of the first member of this equation, are $\frac{\tan \frac{1}{2}(A' - B')}{R}$, and $\frac{R}{\tan \frac{1}{2}(A' + B')}$, respectively; so that the equation manifestly becomes $\frac{\tan \frac{1}{2}(A' - B')}{\tan \frac{1}{2}(A' + B')} = \frac{\sin A' - \sin B'}{\sin A' + \sin B'} \ldots$ (XXII.)

This equation is readily converted into a very useful proportion, viz., *The sum of the sines of two arcs or angles, is to their difference, as the tangent of half the sum of those arcs or angles, is to the tangent of half their difference.*

 26. Operat-

26. Operating with the third and fourth formulæ of the preceding article, as we have already done with the first and second, we shall obtain

$$\frac{\tan\frac{1}{2}(A'+B')\,.\,\tan\frac{1}{2}(A'-B')}{R^2}=\frac{\cos B'-\cos A'}{\cos A'+\cos B'}.$$

In like manner, we have by division,

$$\frac{\sin A'+\sin B'}{\cos A'+\cos B'}=\frac{\sin\frac{1}{2}(A'+B')}{\cos\frac{1}{2}(A'+B')}=\tan\tfrac{1}{2}(A'+B');\ \frac{\sin A'+\sin B'}{\cos B'-\cos A'}=\cot\tfrac{1}{2}(A'-B');$$

$$\frac{\sin A'-\sin B'}{\cos A'+\cos B'}=\tan\tfrac{1}{2}(A'-B')\ldots\frac{\sin A'-\sin B'}{\cos B'-\cos A'}=\cot\tfrac{1}{2}(A'+B'),$$

$$\frac{\cos A'+\cos B'}{\cos B'-\cos A'}=\frac{\cot\frac{1}{2}(A'+B')}{\tan\frac{1}{2}(A'-B')}.$$

Making $B=0$, in one or other of these expressions, there results,

$$\left.\begin{aligned}\frac{\sin A'}{1+\cos A'}&=\tan\tfrac{1}{2}A'=\frac{1}{\cot\frac{1}{2}A'}.\\ \frac{\sin A'}{1-\cos A'}&=\cot\tfrac{1}{2}A'=\frac{1}{\tan\frac{1}{2}A'}.\\ \frac{1+\cos A'}{1-\cos A'}&=\frac{\cot\frac{1}{2}A'}{\tan\frac{1}{2}A'}=\cot^2\tfrac{1}{2}A'=\frac{1}{\tan^2\frac{1}{2}A'}.\end{aligned}\right\}\ (xxii.)$$

These theorems will find their application in some of the investigations of spherical trigonometry.

27. Once more, dividing the expression for $\sin(A\pm B)$ by that for $\cos(A\pm B)$, there results

$$\frac{\sin(A\pm B)}{\cos(A\pm B)}=\frac{\sin A\,.\,\cos B\pm\sin B\,.\,\cos A}{\cos A\,.\,\cos B\mp\sin A\,.\,\sin B}:$$

then dividing both numerator and denominator of the second fraction, by $\cos A\,.\,\cos B$, and recollecting that $\frac{\sin}{\cos}=\frac{\tan}{R}$, we shall thus obtain

$$\frac{\tan(A\pm B)}{R}=\frac{R(\tan A\pm\tan B)}{R^2\mp\tan A\,.\,\tan B};$$

or, lastly, $\tan(A\pm B)=\dfrac{R^2(\tan A\pm\tan B)}{R^2\mp\tan A\,.\,\tan B}$. . . . (XXIII.)

Also, since $\cot=\frac{R^2}{\tan}$, we shall have

$$\cot(A\pm B)=\frac{R^2}{\tan(A\pm B)}=\frac{R^2\mp\tan A\,.\,\tan B}{\tan A\pm\tan B};$$

which, after a little reduction, becomes

$$\cot(A\pm B)=\frac{\cot A\,.\,\cot B\mp R^2}{\cot B\pm\cot A}.\ \ldots\ (XXIV.)$$

28. We might now proceed to deduce expressions for the tangents, cotangents, secants, &c, of multiple arcs, as well as some

some of the usual formulæ of verification in the construction of tables, such as

$$\sin(54^\circ + A) + \sin(54^\circ - A) - \sin(18^\circ + A) - \sin(18^\circ - A) = \sin(90^\circ - A);$$
$$\sin A + \sin(36^\circ - A) + \sin(72^\circ + A) = \sin(36^\circ + A) + \sin(72^\circ - A).$$
&c, &c.

But, as these enquiries would extend this chapter to too great a length, we shall pass them by; and merely investigate a few properties where *more* than two arcs or angles are concerned, and which may be of use in some subsequent parts of this volume.

29. Let A, B, C, be any three arcs or angles, and suppose radius to be unity; then

$$\sin(B + C) = \frac{\sin A \cdot \sin C + \sin B \cdot \sin(A + B + C)}{\sin(A + B)}.$$

For, by equa. v, $\sin(A + B + C) = \sin A \cdot \cos(B + C) + \cos A \cdot \sin(B + C)$, which, (putting $\cos B \cdot \cos C - \sin B \cdot \sin C$ for $\cos(B + C)$), is $= \sin A \cdot \cos B \cdot \cos C - \sin A \cdot \sin B \cdot \sin C + \cos A \cdot \sin(B + C)$; and, multiplying by $\sin B$, and adding $\sin A \cdot \sin C$, there results $\sin A \cdot \sin C + \sin B \cdot \sin(A + B + C) = \sin A \cdot \cos B \cdot \cos C \cdot \sin B + \sin A \cdot \sin C \cdot \cos^2 B + \cos A \cdot \sin B \cdot \sin(B + C) = \sin A \cdot \cos B \cdot (\sin B \cdot \cos C + \cos B \cdot \sin C) + \cos A \cdot \sin B \cdot \sin(B + C) = \sin A \cdot \cos B \cdot \sin(B + C) + \cos A \cdot \sin A \cdot \sin(B + C) = (\sin A \cdot \cos B + \cos A \cdot \sin B) \times \sin(B + C) = \sin(A + B) \cdot \sin(B + C)$. Consequently, by dividing by $\sin(A + B)$, we obtain the expression above given.

In a similar manner it may be shown, that

$$\sin(B - C) = \frac{\sin A \cdot \sin C - \sin B \cdot \sin(A - B + C)}{\sin(A - B)}.$$

30. If A, B, C, D, represent four arcs or angles, then writing $C + D$ for C in the preceding investigation, there will result,

$$\sin(B + C + D) = \frac{\sin A \cdot \sin(C + D) + \sin B \cdot \sin(A + B + C + D)}{\sin(A + B)}.$$

A like process for five arcs or angles will give

$$\sin(B + C + D + E) = \frac{\sin A \cdot \sin(C + D + E) + \sin B \cdot \sin(A + B + C + D + E)}{\sin(A + B)}.$$

And for any number, A, B, C, &c, to L,

$$\sin(B + C + \ldots L) = \frac{\sin A \cdot \sin(C + D + \ldots L) + \sin B \cdot \sin(A + B + C + \ldots L)}{\sin(A + B)}.$$

31. Taking again the three A, B, C, we have

$$\sin(B - C) = \sin B \cdot \cos C - \sin C \cdot \cos B,$$
$$\sin(C - A) = \sin C \cdot \cos A - \sin A \cdot \cos C,$$
$$\sin(A - B) = \sin A \cdot \cos B - \sin B \cdot \cos A.$$

Multiplying the first of these equations by sin A, the second

by sin B, the third by sin C; then adding together the equations thus transformed, and reducing; there will result,

$$\sin A \,.\, \sin(B-C)+\sin B \,.\, \sin(C-A)+\sin C \,.\, \sin(A-B)=0,$$
$$\cos A \,.\, \sin(B-C)+\cos B \,.\, \sin(C-A)+\cos C \,.\, \sin(A-B)=0.$$

These two equations obtaining for any three angles whatever, apply evidently to the three angles of any triangle.

32. Let the series of arcs or angles A, B, C, D . . . . L, be contemplated, then we have (art. 24),

$$\sin(A+B)\,.\,\sin(A-B)=\sin^2 A-\sin^2 B,$$
$$\sin(B+C)\,.\,\sin(B-C)=\sin^2 B-\sin^2 C,$$
$$\sin(C+D)\,.\,\sin(C-D)=\sin^2 C-\sin^2 D.$$
&c. &c. &c.
$$\sin(L+A)\,.\,\sin(L-A)=\sin^2 L-\sin^2 A.$$

If all these equations be added together, the second member of the equation will vanish, and of consequence we shall have

$$\sin(A+B)\,.\,\sin(A-B)+\sin(B+C)\,.\,\sin(B-C)+\&c\ldots$$
$$\ldots\ldots+\sin(L+A)+\sin(L-A)=0.$$

Proceeding in a similar manner with sin (A − B), cos (A + B), sin (B − C), cos (B + C), &c, there will at length be obtained

$$\cos(A+B)\,.\,\sin(A-B)+\cos(B+C)\,.\,\sin(B-C)+\&c\ldots$$
$$\ldots\ldots+\cos(L+A)\,.\,\sin(L-A)=0.$$

33. If the arcs A, B, C, &c . . . . L form an arithmetical progression, of which the first term is 0, the common difference D', and the last term L any number n of circumferences; then will $B-A=D'$, $C-B=D'$, &c, $A+B=2D'$, $B+C=3D'$, &c: and dividing the whole by $\sin D'$, the preceding equations will become

$$\left.\begin{array}{l}\sin D'+\sin 3D'+\sin 5D'+\&c=0,\\ \cos D'+\cos 3D'+\cos 5D'+\&c=0.\end{array}\right\}\text{(XXV.)}$$

If E' were equal $2D'$, these equations would become

$$\sin D'+\sin(D'+E')+\sin(D'+2E')+\sin(D'+3E')+\&c=0,$$
$$\cos D'+\cos(D'+E')+\cos(D'+2E')+\cos(D'+3E')+\&c=0.$$

34. The last equation, however, only shows the sums of sines and cosines of arcs or angles in arithmetical progression, when the common difference is to the first term in the ratio of 2 to 1. To investigate a *general* expression for an infinite series of this kind, let

$$s=\sin A+\sin(A+B)+\sin(A+2B)+\sin(A+3B)+\&c.$$

Then, since this series is a recurring series, whose scale of relation is $2\cos B-1$, it will arise from the developement of a fraction whose denominator is $1-2z\,.\,\cos B+z^2$, making $z=1$.

Now

Now this fraction will be $= \frac{\sin A + z[\sin(A+B) - 2\sin A \cdot \cos B]}{1 - 2z \cdot \cos B + z^2}$.
Therefore, when $z = 1$, we have
$s = \frac{\sin A + \sin(A+B) - 2\sin A \cdot \cos B}{2 - 2\cos B}$; and this, because $2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$ (art. 21), is equal to $\frac{\sin A - \sin(A-B)}{2(1-\cos B)}$. But, since $\sin A' - \sin B' = 2\cos\frac{1}{2}(A'+B') \cdot \sin\frac{1}{2}(A'-B')$, by art. 25, it follows, that $\sin A - \sin(A-B) = 2\cos(A - \frac{1}{2}B)\sin\frac{1}{2}B$; besides which, we have $1 - \cos B = 2\sin^2\frac{1}{2}B$. Consequently the preceding expression becomes $s = \sin A + \sin(A+B) + \sin(A+2B) + \sin(A+3B) + \&c$, *ad infinitum* $= \frac{\cos(A - \frac{1}{2}B)}{2\sin\frac{1}{2}B} \ldots$ (XXVI.)

35. To find the sum of $n+1$ terms of this series, we have simply to consider that the sum of the terms past the $(n+1)$th, that is, the sum of $\sin[A+(n+1)B] + \sin[A+(n+2)B] + \sin[A+(n+3)B] + \&c$, *ad infinitum*, is, by the preceding theorem, $= \frac{\cos[A+(n+\frac{1}{2})B]}{2\sin\frac{1}{2}B}$. Deducting this, therefore, from the former expression, there will remain, $\sin A + \sin(A+B) + \sin(A+2B) + \sin(A+3B) + \ldots \sin(A+nB) = \frac{\cos(A-\frac{1}{2}B) - \cos[A+(n+\frac{1}{2})B]}{2\sin\frac{1}{2}B} = \frac{\sin(A+\frac{1}{2}nB) \cdot \sin\frac{1}{2}(n+1)B}{\sin\frac{1}{2}B}$. (XXVII.)

By like means it will be found, that the sums of the cosines of arcs or angles in arithmetical progression, will be $\cos A + \cos(A+B) + \cos(A+2B) + \cos(A+3B) + \&c$, *ad infinitum* $= -\frac{\sin(A-\frac{1}{2}B)}{2\sin\frac{1}{2}B} \ldots$ (XXVIII.)

Also,

$\cos A + \cos(A+B) + \cos(A+2B) + \cos(A+3B) + \ldots$
$\ldots (\cos A + nB) = \frac{\cos(A+\frac{1}{2}nB) \cdot \sin\frac{1}{2}(n+1)B}{\sin\frac{1}{2}B} \ldots$ (XXIX.)

36. With regard to the tangents of more than two arcs, the following property (the only one we shall here deduce) is a very curious one, which has not yet been inserted in works of Trigonometry, though it has been long known to mathematicians. Let the three arcs A, B, C, together make up the whole circumference, ○: then, since $\tan(A+B) = \frac{R^2(\tan A + \tan B)}{R^2 - \tan A \cdot \tan B}$ (by equa. XXIII), we have $R^2 \times (\tan A + \tan B + \tan C) = R^2 \times [\tan A + \tan B - \tan(A+B)] = R^2 \times (\tan A + \tan B - \frac{R^2(\tan A + \tan B)}{R^2 - \tan A \cdot \tan B}) =$, by actual multiplication and reduction, to $\tan A \cdot \tan B \cdot \tan C$, since $\tan C = \tan[○ - (A+B)] = -\tan(A+B) = -\frac{R^2(\tan A + \tan B)}{R^2 - \tan A \cdot \tan B}$, by what has preceded

preceded in this article. The result therefore is, that *the sum of the tangents of any three arcs which together constitute a circle, multiplied by the square of the radius, is equal to the product of those tangents.* . . . (XXX.)

Since both arcs in the second and fourth quadrants have their tangents considered negative, the above property will apply to arcs any way trisecting a semicircle; and it will therefore apply to the angles of a plane triangle, which are, together, measured by arcs constituting a semicircle. So that, if radius be considered as unity, we shall find that, *the sum of the tangents of the three angles of any plane triangle, is equal to the continued product of those tangents.* (XXXI.)

37. Having thus given the chief properties of the sines, tangents, &c, of arcs, their sines, products, and powers, we shall merely subjoin investigations of theorems for the 2d and 3d cases in the solutions of plane triangles. Thus, with respect to the second case, where two sides and their included angle are given:

By equa IV, $a : b :: \sin A : \sin B$.

By compos. and division $\}$ $a+b : a-b :: \sin A + \sin B : \sin A - \sin B$.

But, eq. XXII, $\tan \frac{1}{2}(A+B) : \tan \frac{1}{2}(A-B) :: \sin A + \sin B : \sin A - \sin B$; whence, ex equal. $a+b : a-b :: \tan \frac{1}{2}(A+B) : \tan \frac{1}{2}(A-B)$. . . . (XXXII.)

Agreeing with the result of the geometrical investigation, at pa. 10, vol. ii.

38. If, instead of having the two sides a, b, given, we know their *logarithms*, as frequently happens in geodesic operations, $\tan \frac{1}{2}(A-B)$ may be readily determined without first finding the number corresponding to the logs. of a and b. For if a and b were considered as the sides of a right-angled triangle, in which ϕ denotes the angle opposite the side a, then would $\tan \phi = \frac{a}{b}$. Now, since a is supposed greater than b, this angle will be greater than half a right angle, or it will be measured by an arc greater than $\frac{1}{8}$ of the circumference, or than $\frac{1}{8}\bigcirc$. Then, because $\tan(\phi - \frac{1}{8}\bigcirc) = \frac{\tan\phi - \tan\frac{1}{8}\bigcirc}{1+\tan\phi\tan\frac{1}{8}\bigcirc}$, and because $\tan \frac{1}{8}\bigcirc = R = 1$, we have

$$\tan(\phi - \tfrac{1}{8}\bigcirc) = (\tfrac{a}{b} - 1) \div (1 + \tfrac{a}{b}) = \frac{a-b}{a+b}.$$

And, from the preceding article,

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{\tan\frac{1}{2}(A-B)}{\cot\frac{1}{2}C} : \text{consequently,}$$

$$\tan \tfrac{1}{2}(A-B) = \cot \tfrac{1}{2}C \cdot \tan(\phi - \tfrac{1}{8}\bigcirc) \ldots \text{(XXXIII.)}$$

From

From this equation we have the following practical rule: Subtract the less from the greater of the given logs, the remainder will be the log tan of an angle: from this angle take 45 degrees, and to the log tan of the remainder add the log cotan of half the given angle; the sum will be the log tan of half the *difference* of the other two angles of the plane triangle.

39. The remaining case is that in which the three sides of the triangle are known, and for which indeed we have already obtained expressions for the angles in arts. 6 and 8. But, as neither of these is best suited for logarithmic computation, (however well fitted they are for instruments of investigation), another may be deduced thus: In the equation for cos A, (given equation II), viz, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, if we substitute, instead of cos A, its value $1 - 2\sin^2 \frac{1}{2}A$, change the signs of all the terms, transpose the 1, and divide by 2, we shall have $\sin^2 \frac{1}{2}A = \frac{a^2 - b^2 - c^2 + 2bc}{4bc} = \frac{a^2 - (b-c)^2}{4bc}$.
Here, the numerator of the second member being the product of the two factors $(a + b - c)$ and $(a - b + c)$, the equation will become $\sin^2 \frac{1}{2}A = \frac{\frac{1}{2}(a+b-c) \cdot \frac{1}{2}(a-b+c)}{bc}$. But, since $\frac{1}{2}(a+b-c) = \frac{1}{2}(a+b+c) - c$, and $\frac{1}{2}(a-b+c) = \frac{1}{2}(a+b+c) - b$; if we put $s = a + b + c$, and extract the square root, there will result,

$$\left.\begin{array}{ll} & \sin \frac{1}{2}A = \sqrt{\frac{(\frac{1}{2}s - b) \cdot (\frac{1}{2}s - c)}{bc}}. \\ \text{In like manner} \Big\} & \sin \frac{1}{2}B = \sqrt{\frac{(\frac{1}{2}s - a) \cdot (\frac{1}{2}s - c)}{ac}}. \\ & \sin \frac{1}{2}C = \sqrt{\frac{(\frac{1}{2}s - a) \cdot (\frac{1}{2}s - b)}{ab}}. \end{array}\right\} \quad \text{(XXXIV.)}$$

These expressions, besides their convenience for logarithmic computation, have the further advantage of being perfectly free from ambiguity, because the half of any angle of a plane triangle will always be *less* than a right angle.

40. The student will find it advantageous to collect into one place all those formulæ which relate to the computation of sines, tangents, &c*; and, in another place, those which are of use in the solutions of plane triangles: the former of these

* What is here given being only a brief sketch of an inexhaustible subject; the reader who wishes to pursue it further is referred to the copious Introduction to our Mathematical Tables, and the comprehensive treatises on Trigonometry, by Emerson and many other modern writers on the same subject, where he will find his curiosity richly gratified.

these are equations V, VIII, IX, X, XI, *x*, *xi*, XII, XIII, XIV, XV, XVI, XVII, XVIII, XIX, XX, XXII, *xxii*, XXIII, XXIV, XXVII; the latter are equa. II, III, IV, VII, XXXII, XXXIII, XXXIV.

To exemplify the use of some of these formulæ, the following exercises are subjoined.

EXERCISES.

Ex. 1. Find the sines and tangents of 15°, 30°, 45°, 60°, and 75°: and show how from thence to find the sines and tangents of several of their submultiples.

First, with regard to the arc of 45°, the sine and cosine are manifestly equal; or they form the perpendicular and base of a right-angled triangle whose hypothenuse is equal to the assumed radius. Thus, if radius be R, the sine and cosine of 45°, will each be $=\sqrt{\frac{1}{2}R^2}=R\sqrt{\frac{1}{2}}=\frac{1}{2}R\sqrt{2}$. If R be equal to 1, as is the case with the tables in use, then

$$\sin 45° = \cos 45° = \tfrac{1}{2}\sqrt{2} = \cdot7071068.$$

$$\tan 45° = \frac{\sin}{\cos} = 1\cdot = \frac{\cos}{\sin} = \text{cotangent } 45°.$$

Secondly, for the sines of 60° and of 30°: since each angle in an equilateral triangle contains 60°, if a perpendicular be demitted from any one angle of such a triangle on the opposite side, considered as a base, that perpendicular will be the sine of 60°, and the half base the sine of 30°, the side of the triangle being the assumed radius. Thus, if it be R, we shall have $\frac{1}{2}R$ for the sine of 30°, and $\sqrt{R^2-\frac{1}{4}R^2}=\frac{1}{2}R\sqrt{3}$, for the sine of 60°. When R = 1, these become

$$\sin 30° = \cdot5 \ldots\ldots \sin 60° = \cos 30° = \cdot8660254.$$

$$\text{Hence, } \tan 30° = \frac{\cdot5}{\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3} = \cdot5773503,$$

$$\tan 60° = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3} = \ldots\ldots 1\cdot7320508.$$

Consequently, tan 60° = 3 tan 30°.

Thirdly, for the sines of 15° and 75°, the former arc is the half of 30°, and the latter is the complement of that half arc. Hence, substituting 1 for R and $\frac{1}{2}\sqrt{3}$, for cos A, in the expression $\sin\frac{1}{2}A = \pm\frac{1}{2}\sqrt{2R^2 \pm 2R\cos A}$ . . . (equa. XII), it becomes $\sin 15° = \frac{1}{2}\sqrt{2-\sqrt{3}} = \cdot2588190$.

Hence, $\sin 75° = \cos 15° = \sqrt{1-\frac{1}{4}(2-\sqrt{3})} = \frac{1}{2}\sqrt{2+\sqrt{3}} = \frac{\sqrt{6}+\sqrt{2}}{4} = \cdot9659258$.

$$\text{Consequently, } \tan 15° = \frac{\sin}{\cos} = \frac{\cdot2588190}{\cdot9659258} = \cdot2679492.$$

$$\text{And, } \tan 75° = \frac{\cdot9659258}{\cdot2588190} = 3\cdot7320508.$$

Now,

Now, from the sine of 30°, those of 6°, 2°, and 1°, may easily be found. For, if $5\text{A} = 30°$, we shall have, from equation x, $\sin 5\text{A} = 5 \sin \text{A} - 20 \sin^3 \text{A} + 16 \sin^5 \text{A}$: or, if $\sin \text{A} = x$, this will become $16x^5 - 20x^3 + 5x = \cdot 5$. This equation solved by any of the approximating rules for such equations, will give $x = \cdot 1045285$, which is the sine of 6°.

Next, to find the sine of 2°, we have again, from equation x, $\sin 3\text{A} = 3 \sin \text{A} - 4 \sin^3 \text{A}$: that is, if x be put for sin 2°, $3x - 4x^3 = \cdot 1045285$. This cubic solved, gives $x = \cdot 0348995 = \sin 2°$.

Then, if $\text{s} = \sin 1°$, we shall, from the second of the equations marked x, have $2\text{s}\sqrt{1 - \text{s}^2} = \cdot 0348995$; whence s is found $= \cdot 0174524 = \sin 1°$.

Had the expression for the sines of bisected arcs been applied successively from sin 15°, to sin 7°30′, sin 3°45′, sin 1°52½′, sin 56¼′, &c, a different series of values might have been obtained: or, if we had proceeded from the quinquisection of 45°, to the trisection of 9°, the bisection of 3°, and so on, a different series still would have been found. But what has been done above, is sufficient to illustrate *this* method. The next example will exhibit a very simple and compendious way of ascending from the sines of smaller to those of larger arcs.

Ex. 2. Given the sine of 1°, to find the sine of 2°, and then the sines of 3°, 4°, 5°, 6°, 7°, 8°, 9°, and 10°, each by a single proportion.

Here, taking first the expression for the sine of a double arc, equa. x, we have $\sin 2° = 2 \sin 1° \sqrt{1 - \sin^2 1°} = \cdot 0348995$.

Then it follows from the rule in equa. xx, that

$\sin 1° : \sin 2° - \sin 1° :: \sin 2° + \sin 1° : \sin 3° = \cdot 0523360$
$\sin 2° : \sin 3° - \sin 1° :: \sin 3° + \sin 1° : \sin 4° = \cdot 0697565$
$\sin 3° : \sin 4° - \sin 1° :: \sin 4° + \sin 1° : \sin 5° = \cdot 0871557$
$\sin 4° : \sin 5° - \sin 1° :: \sin 5° + \sin 1° : \sin 6° = \cdot 1045285$
$\sin 5° : \sin 6° - \sin 1° :: \sin 6° + \sin 1° : \sin 7° = \cdot 1218693$
$\sin 6° : \sin 7° - \sin 1° :: \sin 7° + \sin 1° : \sin 8° = \cdot 1391731$
$\sin 7° : \sin 8° - \sin 1° :: \sin 8° + \sin 1° : \sin 9° = \cdot 1564375$
$\sin 8° : \sin 9° - \sin 1° :: \sin 9° + \sin 1° : \sin 10° = \cdot 1736482$

To check and verify operations like these, the proportions should be changed at certain stages. Thus,

$\sin 1° : \sin 3° - \sin 2° :: \sin 3° + \sin 2° : \sin 5°$,
$\sin 1° : \sin 4° - \sin 3° :: \sin 4° + \sin 3° : \sin 7°$,
$\sin 4° : \sin 7° - \sin 3° :: \sin 7° + \sin 3° : \sin 10°$.

The coincidence of the results of these operations with the analogous results in the preceding, will manifestly establish the correctness of both.

Cor.

Cor. By the same method, knowing the sines of 5°, 10°, and 15°, the sines of 20°, 25°, 35°, 55°, 65°, &c, may be found, each by a single proportion. And the sines of 1°, 9°, and 10°, will lead to those of 19°, 29°, 39°, &c. So that the sines may be computed to any arc: and the tangents and other trigonometrical lines, by means of the expressions in art. 4, &c.

Ex. 3. Find the sum of all the natural sines to every minute in the quadrant, radius = 1.

In this problem the actual addition of all the terms would be a most tiresome labour: but the solution by means of equation XXVII, is rendered very easy. Applying that theorem to the present case, we have $\sin(\text{A} + \frac{1}{2}n\text{B}) = \sin 45°$, $\sin \frac{1}{2}(n+1)\text{B} = \sin 45°0'30''$, and $\sin \frac{1}{2}\text{B} = \sin 30''$. Therefore $\frac{\sin 45° \times \sin 45° 0' 30''}{\sin 30''} = 3438{\cdot}2467465$ the sum required.

From another method, the investigation of which is omitted here, it appears that the same sum is equal to $\frac{1}{2}(\cot 30'' + 1)$.

Ex. 4. Explain the method of finding the *logarithmic* sines, cosines, tangents, secants, &c, the natural sines, cosines, &c, being known.

The *natural* sines and cosines being computed to the radius unity, are all proper fractions, or quantities less than unity, so that their logarithms would be negative. To avoid this, the tables of logarithmic sines, cosines, &c, are computed to a radius of 10000000000, or 10^{10}; in which case the logarithm of the radius is 10 times the log of 10, that is, it is 10.

Hence, if s represent any sine to radius 1, then $10^{10} \times s =$ sine of the same arc or angle to rad 10^{10}. And this, in logs is, $\log 10^{10} s = 10 \log 10 + \log s = 10 + \log s$.

The log cosines are found by the same process, since the cosines are the sines of the complements.

The logarithmic expressions for the tangents, &c, are deduced thus:

Tan $= \text{rad}\frac{\sin}{\cos}$. Theref. $\log \tan = \log \text{rad} + \log \sin - \log \cos = 10 + \log \sin - \log \cos$.

Cot $= \frac{\text{rad}^2}{\tan}$. Therf. $\log \cot = 2 \log \text{rad} - \log \tan = 20 - \log \tan$.

Sec $= \frac{\text{rad}^2}{\cos}$. Therf. $\log \sec = 2 \log \text{rad} - \log \cos = 20 - \log \cos$.

Cosec $= \frac{\text{rad}^2}{\sin}$. Therf. $\text{l.}\,\text{cosec} = 2 \log \text{rad} - \log \sin = 20 - \log \sin$.

Versed sine $= \frac{\text{ch'd ord}^2}{\text{diam}} = \frac{(2 \sin \frac{1}{2} \text{arc})^2}{2 \text{ rad}} = \frac{2 \times \sin^2 \frac{1}{2} \text{arc}}{\text{rad}}$.

Therefore, $\log \text{vers} \sin = \log 2 + 2 \log \sin \frac{1}{2} \text{arc} - 10$.

Ex. 5.

Ex. 5. Given the sum of the natural tangents of the angles A and B of a plane triangle $= 3 \cdot 1601988$, the sum of the tangents of the angles B and C $= 3 \cdot 8765577$, and the continued product, tan A . tan B . tan C $= 5 \cdot 3047057$: to find the angles A, B, and C.

It has been demonstrated in art. 36, that when radius is unity, the product of the natural tangents of the three angles of a plane triangle is equal to their continued product. Hence the process is this:

$$\begin{array}{lcl} \text{From } \tan A + \tan B + \tan C & = & 5 \cdot 3047057 \\ \text{Take } \tan A + \tan B \ldots\ldots & = & 3 \cdot 1601988 \\ \hline \text{Remains } \tan C \ldots\ldots\ldots & = & 2 \cdot 1445069 = \tan 65^\circ. \\ \text{From } \tan A + \tan B + \tan C & = & 5 \cdot 3047057 \\ \text{Take } \tan B + \tan C \ldots\ldots & = & 3 \cdot 8765577 \\ \hline \text{Remains } \tan A \ldots\ldots\ldots & = & 1 \cdot 4281480 = \tan 55^\circ. \end{array}$$

Consequently, the three angles are 55°, 60°, and 65°.

Ex. 6. There is a plane triangle, whose sides are three consecutive terms in the natural series of integer numbers, and whose largest angle is just double the smallest. Required the sides and angles of that triangle?

If A, B, C, be three angles of a plane triangle, a, b, c, the sides respectively opposite to A, B, C; and $s = a + b + c$. Then from equa. III and XXXIV, we have

$$\sin A = \frac{2}{bc}\sqrt{\tfrac{1}{2}s(\tfrac{1}{2}s - a) \cdot (\tfrac{1}{2}s - b) \cdot (\tfrac{1}{2}s - c)},$$

$$\text{and } \sin \tfrac{1}{2}C = \sqrt{\frac{(\frac{1}{2}s - a)(\frac{1}{2}s - b)}{ab}}.$$

Let the three sides of the required triangle be represented by x, $x + 1$, and $x + 2$; the angle A being supposed opposite to the side x, and C opposite to the side $x + 2$: then the preceding expressions will become

$$\sin A = \frac{2}{(x+1) \cdot (x+2)}\sqrt{\frac{3x+3}{2} \cdot \frac{x+3}{2} \cdot \frac{x+1}{2} \cdot \frac{x-1}{2}}.$$

$$\sin \tfrac{1}{2}C = \sqrt{\frac{(x+1) \cdot (x+3)}{4 \cdot (x+1)}}.$$

Assuming these two expressions equal to each other, as they ought to be, by the question; there results, after a little reduction, $\sqrt{\frac{1}{x}} = \frac{\sqrt{3(x-1)}}{x+2}$, or $3x(x-1) = (x+2)^2$, an equation whose root is 4 or $-\frac{1}{2}$. Hence 4, 5, and 6, are the sides of the triangle.

$\sin A = \frac{2}{5 \cdot 6}\sqrt{\frac{15}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2}} = \frac{2}{5 \cdot 6}\sqrt{\frac{15}{4} \cdot \frac{15}{4} \cdot 7} = \frac{2 \cdot 15}{5 \cdot 6 \cdot 4}\sqrt{7} = \frac{1}{4}\sqrt{7}$.

$\sin B = \frac{5}{16}\sqrt{7}$; $\sin C = \frac{6}{16}\sqrt{7}$; $\sin \frac{1}{2}C = \sqrt{\frac{7 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 5}} = \frac{1}{4}\sqrt{7}$.

The angles are, A $= 41^\circ \cdot 409603 = 41^\circ 24' 34'' 34'''$,
B $= 55^\circ \cdot 771191 = 55\ 46\ 16\ 18$,
C $= 82^\circ \cdot 819206 = 82\ 49\ 9\ 8$

Ex. 7.

Ex. 7. Demonstrate that $\sin 18^\circ = \cos 72^\circ$ is $= \frac{1}{4}R(-1+\sqrt{5})$, and $\sin 54^\circ = \cos 36^\circ$ is $= \frac{1}{4}R(1+\sqrt{5})$.

Ex. 8. Demonstrate that the sum of the sines of two arcs which together make 60°, is equal to the sine of an arc which is greater than 60° by either of the two arcs: Ex. gr. $\sin 3' + \sin 59^\circ 57' = \sin 60^\circ 3'$; and thus that the tables may be continued by addition only.

Ex. 9. Show the truth of the following proportion: As the sine of half the difference of two arcs, which together make 60°, or 90°, respectively, is to the difference of their sines; so is 1 to $\sqrt{3}$, or $\sqrt{2}$, respectively.

Ex. 10. Demonstrate that the sum of the squares of the sine and versed sine of an arc, is equal to the square of double the sine of half the arc.

Ex. 11. Demonstrate that the sine of an arc is a mean proportional between half the radius and the versed sine of double the arc.

Ex. 12. Show that the secant of an arc is equal to the sum of its tangent and the tangent of half its complement.

Ex. 13. Prove that, in any plane triangle, the base is to the difference of the other two sides, as the sine of half the sum of the angles at the base, to the sine of half their difference: also, that the base is to the sum of the other two sides, as the cosine of half the sum of the angles at the base, to the cosine of half their difference.

Ex. 14. How must three trees, A, B, C, be planted, so that the angle at A may be double the angle at B, the angle at B double that at C; and so that a line of 400 yards may just go round them? Ans. AB [illegible]

Ex. 15. In a certain triangle, the sines of the three angles are as the numbers 17, 15, and 8, and the perimeter is 160. What are the sides and angles?

Ex. 16. The logarithms of two sides of a triangle are 2·2407293 and 2·5378191, and the included angle, is 37°20′. It is required to determine the other angles, without first finding any of the sides?

Ex. 17. The sides of a triangle are to each other as the fractions 1, $\frac{1}{4}$, $\frac{1}{5}$: what are the angles?

Ex. 18. Show that the secant of 60°, is double the tangent of 45°, and that the secant of 45° is a mean proportional between the tangent of 45° and the secant of 60°.

Ex. 19. Demonstrate that 4 times the rectangle of the sines of two arcs, is equal to the difference of the squares of the chords of the sum and difference of those arcs.

Ex. 20. Convert the equations marked XXXIV into their equivalent logarithmic expressions; and by means of them and equa. IV, find the angles of a triangle whose sides are 5, 6, and 7.

Ex. 21. Find the arc whose tangent and cotangent shall together be equal to 4 times the radius.

Ex. 22. Find the arc whose sine added to its cosine shall be equal to a; and show the limits of possibility.

Ex. 23. Find the arc whose secant and cotangent shall be equal.

CHAPTER IV.

SPHERICAL TRIGONOMETRY.

SECTION I.

General Properties of Spherical Triangles.

ART. 1. *Def.* 1. Any portion of a spherical surface bounded by three arcs of great circles, is called a *Spherical Triangle.*

Def. 2. Spherical Trigonometry is the art of computing the measures of the sides and angles of spherical triangles.

Def. 3. A *right-angled* spherical triangle has one right angle: the sides about the right angle are called *legs;* the side opposite to the right angle is called the *hypothenuse.*

Def. 4. A *quadrantal* spherical triangle has one side equal to 90° or a quarter of a great circle.

Def. 5. Two arcs or angles, when compared together, are said to be *alike*, or of the *same affection*, when both are less than 90°, or both are greater than 90°. But when one is greater and the other less than 90°, they are said to be *unlike*, or of *different affections.*

ART. 2. The small circles of the sphere do not fall under consideration in Spherical Trigonometry; but such only as have the same centre with the sphere itself. And hence it is that

that spherical trigonometry is of so much use in Practical Astronomy, the apparent heavens assuming the shape of a concave sphere, whose centre is the same as the centre of the earth.

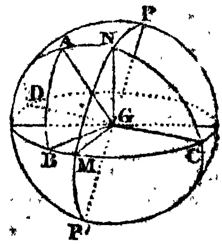
3. Every spherical triangle has three sides and three angles: and if any three of these six parts, be given, the remaining three may be found, by some of the rules which will be investigated in this chapter.

4. In *plane* trigonometry, the knowledge of the three angies is not sufficient for ascertaining the sides: for in that case the *relations* only of the three sides can be obtained, and not their absolute values: whereas, in *spherical* trigonometry, where the sides are circular arcs, whose values depend on their proportion to the whole circle, that is, on the number of degrees they contain, the sides may always be determined when the three angles are known. Other remarkable differences between plane and spherical triangles are, 1st. That in the former, two angles always determine the third; while in the latter they never do. 2dly. The surface of a plane triangle cannot be determined from a knowledge of the angles alone; while that of a spherical triangle always can.

5. The *sides* of a spherical triangle are all arcs of great circles, which, by their intersection on the surface of the sphere, constitute that triangle.

6. The *angle* which is contained between the arcs of two great circles, intersecting each other on the surface of the sphere, is called a spherical angle; and its measure is the same as the measure of the plane angle which is formed by two lines issuing from the same point of, and perpendicular to, the common section of the planes which determine the containing sides: that is to say, it is the same as the angle made by those planes. Or, it is equal to the plane angle formed by the tangents to those arcs at their point of intersection.

7. Hence it follows, that the surface of a spherical triangle BAC, and the three planes which determine it, form a kind of triangular pyramid, BCGA, of which the vertex G is at the centre of the sphere, the base ABC a portion of the spherical surface, and the faces AGC, AGB, BGC, sectors of the great circles whose intersections determine the sides of the triangle.



Def. 6. A line perpendicular to the plane of a great circle, passing through the centre of the sphere, and terminated by

two

two points, diametrically opposite, at its surface, is called the *axis* of such circle; and the extremities of the axis, or the points where it meets the surface, are called the *poles* of that circle. Thus, PGP′ is the axis, and P, P′, are the poles, of the great circle CND.

If we conceive any number of less circles, each parallel to the said great circle, this axis will be perpendicular to them likewise; and the points P, P′, will be their poles also.

8. Hence, each pole of a great circle is 90° distant from every point in its circumference; and all the arcs drawn from either pole of a little circle to its circumference, are equal to each other.

9. It likewise follows, that all the arcs of great circles drawn through the poles of another great circle, are perpendicular to it: for, since they are great circles by the supposition, they all pass through the centre of the sphere, and consequently through the axis of the said circle. The same thing may be affirmed with regard to small circles.

10. Hence, in order to find the *poles* of any circle, it is merely necessary to describe, upon the surface of the sphere, two great circles perpendicular to the plane of the former; the points where these circles intersect each other will be the poles required.

11. It may be inferred also, from the preceding, that if it were proposed to draw, from any point assumed on the surface of the sphere, an arc of a circle which may measure the shortest distance from that point, to the circumference of any given circle; this arc must be so described, that its prolongation may pass through the poles of the given circle. And conversely, if an arc pass through the poles of a given circle, it will measure the shortest distance from any assumed point to the circumference of that circle.

12. Hence again, if upon the sides, AC and BC, (produced if necessary) of a spherical triangle BCA, we take the arcs CN, CM, each equal 90°, and through the radii GN, GM (figure to art. 7) draw the plane NGM, it is manifest that the point C will be the pole of the circle coinciding with the plane NGM: so that, as the lines GM, GN, are both perpendicular to the common section GC, of the planes AGC, BGC, they measure, by their inclination, the angle of these planes; or the arc NM measures that angle, and consequently the spherical angle BCA.

13. It is also evident that every arc of a little circle, described from the pole C as centre, and containing the same number of degrees as the arc MN, is equally proper for measuring

suring the angle BCA; though it is customary to use only arcs of great circles for this purpose.

14. Lastly, we infer, that if a spherical angle be a right angle, the arcs of the great circles which form it, will pass mutually through the poles of each other: and that, if the planes of two great circles contain each the axis of the other, or pass through the poles of each other, the angle which they include is a right angle.

These obvious truths being premised and comprehended, the student may pass to the consideration of the following theorems.

THEOREM I.

Any Two Sides of a Spherical Triangle are together Greater than the Third.

This proposition is a necessary consequence of the truth, that the shortest distance between any two points, measured on the surface of the sphere, is the arc of a great circle passing through these points.

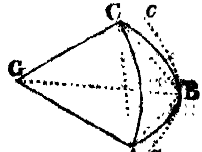
THEOREM II.

The Sum of the Three Sides of any Spherical Triangle is Less than 360 degrees.

For, let the sides AC, BC, (fig. to art. 7) containing any angle A, be produced till they meet again in D: then will the arcs DAC, DBC, be each 180°, because all great circles cut each other into two equal parts: consequently $DAC + DBC = 360°$. But (theorem 1) DA and DB are together greater than the third side AB of the triangle DAB; and therefore, since $CA + CB + DA + DB = 360°$, the sum $CA + CB + AB$ is less than 360°. Q. E. D.

THEOREM III.

The Sum of the Three Angles of any Spherical Triangle is always Greater than Two Right Angles, but Less than Six.



For, let ABC be a spherical triangle, G the centre of the sphere, and let the chords of the arcs AB, BC, AC, be drawn: these chords constitute a rectilinear triangle, the sum of whose three angles is equal to two right angles. But the angle at B made by the chords AB, BC, is *less* than the angle *a*B*c*, formed by the two tangents B*a*, B*c*, or less than the angle of inclination

inclination of the two planes GBC, GBA, which (art. 6) is the spherical angle at B; consequently the spherical angle at B is *greater* than the angle at B made by the chords AB, CB. In like manner, the spherical angles at A and C, are greater than the respective angles made by the chords meeting at those points. Consequently, the sum of the three angles of the spherical triangle ABC, is greater than the sum of the three angles of the rectilinear triangle made by the chords AB, BC, AC, that is, greater than two right angles. Q. E. 1°D.

2. The angle of inclination of no two of the planes can be so great as two right angles; because, in that case, the two planes would become but one coutinued plane, and the arcs, instead of being arcs of distinct circles, would be joint arcs of one and the same circle. Therefore, each of the three spherical angles must be less than two right angles; and consequently their sum less than six right angles, Q. E. 2°D.

Cor. 1. Hence it follows, that a spherical triangle may have all its angles either right or obtuse; and therefore the knowledge of any two angles is not sufficient for the determination of the third.

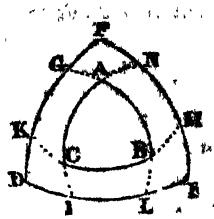
Cor. 2. If the three angles of a spherical triangle be right or obtuse, the three sides are likewise each equal to, or greater than 90°: and, if each of the angles be acute, each of the sides is also less than 90°; and conversely.

Scholium. From the preceding theorem the student may clearly perceive what is the essential difference between plane and spherical triangles, and how absurd it would be to apply the rules of plane trigonometry to the solution of cases in spherical trigonometry. Yet, though the difference between the two kinds of triangles be really so great, still there are various properties which are common to both, and which may be demonstrated exactly in the same manner. Thus, for example, it might be demonstrated here, (as well as with regard to plane triangles in the elements of Geometry, vol. 1) that two spherical triangles are equal to each other, 1st. When the three sides of the one are respectively equal to the three sides of the other. 2dly. When each of them has an equal angle contained between equal sides: and, 3dly. When they have each two equal angles at the extremities of equal bases. It might also be shown, that a spherical triangle is equilateral, isosceles, or scalene, according as it hath three equal, two equal, or three unequal angles: and again, that the greatest side is always opposite to the greatest angle, and the least side to the least angle. But the brevity that our plan requires,

compels us merely to *mention* these particulars. It may be added, however, that a spherical triangle may be at once *right-angled* and *equilateral*; which can never be the case with a plane triangle.

THEOREM IV.

If from the Angles of a Spherical Triangle, as Poles, there be described, on the Surface of the Sphere, Three Arcs of Great Circles, which by their Intersections form another Spherical Triangle; Each Side of this New Triangle will be the Supplement to the Measure of the Angle which is at its Pole, and the Measure of each of its Angles the Supplement to that Side of the Primitive Triangle to which it is Opposite.



From B, A, and C, as poles, let the arcs DF, DE, FE, be described, and by their intersections form another spherical triangle DEF; either side, as DE, of this triangle, is the supplement of the measure of the angle A at its pole; and either angle, as D, has for its measure the supplement of the side AB.

Let the sides AB, AC, BC, of the primitive triangle, be produced till they meet those of the triangle DEF, in the points I, L, M, N, G, K: then, since the point A is the pole of the arc DILE, the distance of the points A and E (measured on an arc of a great circle) will be 90°; also, since C is the pole of the arc EF, the points C and E will be 90° distant: consequently (art. 8) the point E is the pole of the arc AC. In like manner it may be shown, that F is the pole of BC, and D that of AB.

This being premised, we shall have DL $= 90°$, and IE $= 90°$; whence DL + IE = DL + EL + IL = DE + IL = 180°. Therefore DE = 180° − IL: that is, since IL is the measure of the angle BAC, the arc DE is = the supplement of that measure. Thus also may it be demonstrated that EF is equal the supplement to MN, the measure of the angle BCA, and that DF is equal the supplement to GK, the measure of the angle ABC: which constitutes the first part of the proposition.

2dly. The respective measures of the angles of the triangle DEF are supplemental to the opposite sides of the triangles ABC. For, since the arcs AL and BG are each 90°, therefore is AL + BG = GL + AB = 180°; whence GL = 180° − AB; that is, the measure of the angle D is equal to the supplement to AB. So likewise may it be shown that AC, BC, are equal

to

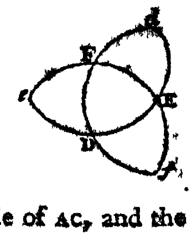
to the supplements to the measures of the respectively opposite angles E and F. Consequently, the measures of the angles of the triangle DEF are supplemental to the several opposite sides of the triangle ABC. Q. E. D.

Cor. 1. Hence these two triangles are called *supplemental* or *polar* triangles.

Cor. 2. Since the three sides DE, EF, DF, are supplements to the measures of the three angles A, B, C; it results that $DE + EF + DF + A + B + C = 3 \times 180° = 540°$. But (th. 2), $DE + EF + DF < 360°$: consequently $A + B + C > 180°$. Thus the first part of theorem 3 is very compendiously demonstrated.

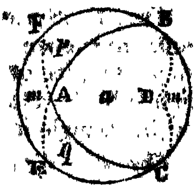
Cor. 3. This theorem suggests mutations that are sometimes of use in computation.—Thus, if three angles of a spherical triangle are given, to find the sides: the student may subtract each of the angles from 180°, and the three remainders will be the three sides of a new triangle; the angles of this new triangle being found, if their measures be each taken from 180°, the three remainders will be the respective sides of the primitive triangle, whose angles were given.

Scholium. The invention of the preceding theorem is due to *Philip Langsberg*. Vide, Simon Stevin, liv. 3, de la Cosmographie, prop. 31 and Alb. Girard in loc. It is often however treated very loosely by authors on trigonometry: some of them speaking of sides as the supplements of angles, and scarcely any of them remarking which of the several triangles formed by the intersection of the arcs DE, EF, DF, is the one in question. Besides the triangle DEF, three others may be formed by the intersection of the semicircles, and if the *whole* circles be considered, there will be *seven* other triangles formed. But the proposition only obtains with regard to the central triangle (of each hemisphere), which is distinguished from the three others in this, that the two angles A and F are situated on the same side of BC, the two B and E on the same side of AC, and the two C and D on the same side of AB.



THEOREM V.

In Every Spherical Triangle the following proportion obtains; viz, As Four Right Angles (or 360°) to the Surface of a Hemisphere; or, as Two Right Angles (or 180°) to a Great Circle of the Sphere; so is the Excess of the three angles of the triangle above Two Right Angles, to the Area of the triangle.



Let ABC be the spherical triangle. Complete one of its sides as BC into the circle BCEF, which may be supposed to bound the upper hemisphere. Prolong also, at both ends, the two sides AB, AC, until they form semicircles estimated from each angle, that is, until BAE = ABD = CAF = ACD = 180°. Then will CBF = 180° = BCE; and consequently the triangle AEF, on the anterior hemisphere, will be equal to the triangle BCD on the opposite hemisphere. Putting m, m', to represent the surface of these triangles, p for that of the triangle BAF, q for that of CAE, and a for that of the proposed triangle ABC. Then a and m' together (or their equal a and m together) make up the surface of a spheric lune comprehended between the two semicircles ACD, ABD, inclined in the angle A: a and p together make up the lune included between the semicircles CAF, CBF, making the angle C: a and q together make up the spheric lune included between the simicircles BCE, BAE, making the angle B. And the surface of each of these lunes, is to that of the hemisphere, as the angle made by the comprehending semicircles, to two right angles. Therefore, putting $\frac{1}{2}$S for the surface of the hemisphere, we have

$$180° : \text{A} :: \tfrac{1}{2}\text{S} : a + m,$$
$$180° : \text{B} :: \tfrac{1}{2}\text{S} : a + q,$$
$$180° : \text{C} :: \tfrac{1}{2}\text{S} : a + p.$$

Whence, $180° : \text{A}+\text{B}+\text{C} :: \frac{1}{2}\text{S} : 3a + m + p + q = 2a + \frac{1}{2}\text{S}$;
and consequently, by division of proportion,

as $180° : \text{A} + \text{B} + \text{C} - 180° :: \frac{1}{2}\text{S} : 2a + \frac{1}{2}\text{S} - \frac{1}{2}\text{S} = 2a$;

or, $180° : \text{A} + \text{B} + \text{C} - 180° :: \frac{1}{4}\text{S} : a = \text{S} . \frac{\text{A}+\text{B}+\text{C}-180°}{360°}$,

Q. E. D*.

Cor. 1. Hence the excess of the three angles of any spherical triangle above two right angles, termed technically the *spherical excess*, furnishes a correct measure of the surface of that triangle.

Cor. 2. If $\pi = 3{\cdot}141593$, and d the diameter of the sphere, then is $\pi d^2 . \frac{\text{A}+\text{B}+\text{C}-180°}{720°}$ = the area of the spherical triangle.

* This determination of the area of a spherical triangle is due to *Albert Girard* (who died about 1632). But the demonstration now commonly given of the rule was first published by Dr. Wallis. It was considered as a mere speculative truth, until General Roy, in 1787, employed it very judiciously in the great Trigonometrical Survey, to correct the errors of spherical angles. See Phil. Trans. vol. 80, and the next chapter of this volume.

Cor. 3.

Cor. 3. Since the length of the radius, in any circle, is equal to the length of 57·2957795 degrees, measured on the circumference of that circle; if the *spherical excess* be multiplied by 57·2957795, the product will express the surface of the triangle in square degrees.

Cor. 4. When $a = 0$, then $A + B + C = 180°$: and when $a = \frac{1}{2}c$, then $A + B + C = 540°$. Consequently the sum of the three angles of a spherical triangle, is always between 2 and 6 right angles: which is another confirmation of th. 3.

Cor. 5. When *two* of the angles of a spherical triangle are right angles, the surface of the triangle varies with its third angle. And when a spherical triangle has *three* right angles its surface is one-eighth of the surface of the sphere.

Remark. Some of the uses of the spherical excess, in the more extensive geodesic operations, will be shown in the following chapter. The mode of finding it, and thence the area when the three angles of a spherical triangle are given, is obvious enough; but it is often requisite to ascertain it by means of other data, as, when two sides and the included angle are given, or when all the three sides are given. In the former case, let a and b be the two sides, C the included angle, and E the spherical excess: then is $\cot \frac{1}{2}E = \frac{\cot \frac{1}{2}a \,.\, \cot \frac{1}{2}b + \cos C}{\sin C}$. When the three sides a, b, c, are given, the spherical excess may be found by the following very elegant theorem, discovered by Simon Lhuillier:

$$\tan \tfrac{1}{4}E = \surd\left(\tan\frac{a+b+c}{4} \,.\, \tan\frac{a+b-c}{4} \,.\, \tan\frac{a-b+c}{4} \,.\, \tan\frac{-a+b+c}{4}\right).$$

The investigation of these theorems would occupy more space than can be allotted to them in the present volume.

THEOREM VI.

In every Spherical Polygon, or surface included by any number of intersecting great circles, the subjoined proportion obtains, viz, As Four Right Angles, or 360°, to the Surface of a Hemisphere; or, as Two Right Angles, or 180°, to a Great Circle of the Sphere; so is the Excess of the Sum of the Angles above the Product of 180° and Two Less than the Number of Angles of the spherical polygon, to its Area.

For, if the polygon be supposed to be divided into as many triangles as it has sides, by great circles drawn from all the angles through any point within it, forming at that point the vertical angles of all the triangles. Then, by th. 5, it will be

as

as $360° : \frac{1}{2}s :: A + B + C - 180° :$ its area. Therefore, putting P for the sum of all the angles of the polygon, n for their number, and V for the sum of all the vertical angles of its constituent triangles, it will be, by composition,
as $360° : \frac{1}{2}s :: P + V - 180°n :$ surface of the polygon. But V is manifestly equal to $360°$ or $180° \times 2$. Therefore, as $360° : \frac{1}{2}s :: P - (n-2)\,180° : \frac{1}{2}s \cdot \frac{P-(n-2)180°}{360°}$, the area of the polygon. Q. E. D.

Cor. 1. If π and d represent the same quantities as in theor. 5 cor. 2, then the surface of the polygon will be expressed by $\pi d^2 \cdot \frac{P-(n-2)180°}{720°}$.

Cor. 2. If $R° = 57{\cdot}2957795$, then will the surface of the polygon in square degrees be $= R^2 \cdot (P-(n-2)180°)$.

Cor. 3. When the surface of the polygon is 0, then $P = (n-2)\,180°$; and when it is a maximum, that is, when it is equal to the surface of the hemisphere, then $P = (n-2)180° + 360° = n \cdot 180°$: Consequently P, the sum of all the angles of any spheric polygon, is always *less* than $2n$ right angles, but *greater* than $(2n-4)$ right angles, n denoting the number of angles of the polygon.

GENERAL SCHOLIUM.

On the Nature and Measure of Solid Angles.

A *Solid angle* is defined by Euclid, that which is made by the meeting of more than two plane angles, which are not in the same plane, in one point.

Others define it the angular space comprized between several planes meeting in one point.

It may be defined still more generally, the *angular space* included between several plane surfaces or one or more curved surfaces, meeting in the point which forms the summit of the angle.

According to this definition, solid angles bear just the same relation to the surfaces which comprize them, as plane angles do to the lines by which they are included: so that, as in the latter, it is not the magnitude of the lines, but their mutual inclination, which determines the angle; just so, in the former it is not the magnitude of the planes, but *their* mutual inclinations which determine the angles. And hence all those geometers, from the time of Euclid down to the present period, who have confined their attention principally to the magnitude

have never been able to develope the properties of this class of geometrical quantities; but have affirmed that no solid angle can be said to be the half or the double of another, and have spoken of the bisection and trisection of solid angles, even in the simplest cases, as impossible problems.

But all this supposed difficulty vanishes, and the doctrine of solid angles becomes simple, satisfactory, and universal in its application, by assuming *spherical surfaces* for their measure; just as circular arcs are assumed for the measures of plane angles*. Imagine, that from the summit of a solid angle (formed by the meeting of three planes) as a centre, any sphere be described, and that those planes are produced till they cut the surface of the sphere; then will the surface of the spherical triangle, included between those planes, be a proper measure of the solid angle made by the planes at their common point of meeting: for no change can be conceived in the relative position of those planes, that is, in the magnitude of the solid angle, without a corresponding and proportional mutation in the surface of the spherical triangle. If, in like manner, the three or more surfaces, which by their meeting constitute another solid angle, be produced till they cut the surface of the same or an equal sphere, whose centre coincides with the summit of the angle; the surface of the spheric triangle or polygon, included between the planes which determine the angle, will be a correct measure of *that* angle. And the ratio which subsists between the areas of the spheric triangles, polygons, or other surfaces thus formed, will be accurately the ratio which subsists between the solid angles, constituted by the meeting of the several planes or surfaces, at the centre of the sphere.

* It may be proper to anticipate here the only objection which *can* be made to this assumption; which is founded on the principle, *that quantities should always be measured by quantities of the same kind.* But this, often and positively as it is affirmed, is by no means necessary; nor in many cases is it possible. To measure is to *compare* mathematically: and if by comparing two quantities, whose ratio we know or can ascertain, with two other quantities whose ratio we wish to know, the point in question becomes determined: it signifies not at all whether the magnitudes which constitute one ratio, are like or unlike the magnitudes which constitute the other ratio. It is thus that mathematicians, with perfect safety and correctness, make use of space as a measure of velocity, mass as a measure of inertia, mass and velocity conjointly as a measure of force, space as a measure of time, weight as a measure of density, expansion as a measure of heat, a certain function of planetary velocity as a measure of distance from the central body, arcs of the same circle as measures of plane angles; and it is in conformity with this general procedure that we adopt surfaces, of the same sphere, as measures of solid angles.

Hence,

Hence, the comparison of solid angles becomes a matter of great ease and simplicity: for, since the areas of spherical triangles are measured by the excess of the sums of their angles each above two right angles (th. 5); and the areas of spherical polygons of n sides, by the excess of the sum of their angles above $2n-4$ right angles (th. 6), it follows, that the magnitude of a trilateral solid angle, will be measured by the excess of the sum of the three angles, made respectively by its bounding planes, above 2 right angles; and the magnitudes of solid angles formed by n bounding planes, by the excess of the sum of the angles of inclination of the several planes above $2n-4$ right angles.

As to solid angles limited by curve surfaces, such as the angles at the vertices of cones; they will manifestly be measured by the spheric surfaces cut off by the prolongation of their bounding surfaces, in the same manner as angles determined by planes are measured by the triangles or polygons, they mark out upon the same, or an equal sphere. In all cases, the maximum limit of solid angles, will be the *plane* towards which the various planes determining such angles approach, as they diverge further from each other about the same summit: just as a right line is the maximum limit of plane angles, being formed by the two bounding lines when they make an angle of 180°. The maximum limit of solid angles is measured by the surface of a hemisphere, in like manner as the maximum limit of plane angles is measured by the arc of a semicircle. The solid right angle (either angle, for example, of a cube) is $\frac{1}{4}(=\frac{1}{2}^2)$ of the maximum solid angle: while the plane right angle is half the maximum plane angle.

The analogy between plane and solid angles being thus traced, we may proceed to exemplify this theory by a few instances; assuming 1000 as the numeral measure of the maximum solid angle $=$ 4 times 90° solid $=$ 360° solid.

1. The solid angles of right prisms are compared with great facility. For, of the three angles made by the three planes which, by their meeting, constitute every such solid angle, two are right angles; and the third is the same as the corresponding plane angle of the polygonal base; on which, therefore, the measure of the solid angle depends. Thus, with respect to the right prism with an equilateral triangular base, each solid angle is formed by planes which respectively make angles of 90°, 90°, and 60°. Consequently $90°+90°+60°-180°=60°$, is the measure of such angle, compared with 360° the maximum angle. It is, therefore, one-sixth of the maximum angle. A right prism with a square base, has, in like manner,

manner, each solid angle measured by $90°+90°+90°-180° = 90°$, which is $\frac{1}{4}$ of the maximum angle. And thus it may be found, that each solid angle of a right prism, with an equilateral

triangular base	is $\frac{1}{6}$ max. angle	$=$	$\frac{1}{6}$ .1000.
square base	is $\frac{1}{4}$	$=$	$\frac{2}{8}$ .1000.
pentagonal base	is	$=$	$\frac{3}{10}$ .1000.
hexagonal	is $\frac{1}{3}$	$=$	$\frac{4}{12}$ .1000.
heptagonal	is	$=$	$\frac{5}{14}$ .1000.
octagonal	is $\frac{3}{8}$	$=$	$\frac{6}{16}$ .1000.
nonagonal	is	$=$	$\frac{7}{18}$ .1000.
decagonal	is $\frac{2}{5}$	$=$	$\frac{8}{20}$ .1000.
undecagonal	is	$=$	$\frac{9}{22}$ .1000.
duodecagonal	is $\frac{5}{12}$	$=$	$\frac{10}{24}$ .1000.
m gonal	is . . .	$=$	$\frac{m-2}{2m}$ .1000.

Hence it may be deduced, that each solid angle of a regular prism, with triangular base, is *half* each solid angle of a prism with a regular hexagonal base. Each with regular

square base	$= \frac{2}{3}$ of each,	with regular octagonal base,	
pentagonal	$= \frac{3}{4}$	. . . .	decagonal,
hexagonal	$= \frac{4}{5}$	. . . .	duodecagonal,
$\frac{1}{2}m$ gonal	$= \frac{m-4}{m-2}$	. . . .	m gonal base.

Hence again we may infer, that the sum of all the solid angles of any prism of triangular base, whether that base be regular or irregular, is *half* the sum of the solid angles of a prism of quadrangular base, regular or irregular. And, the sum of the solid angles of any prism of

tetragonal base is	$= \frac{2}{3}$	sum of angles in prism of	pentag. base,
pentagonal . . .	$= \frac{3}{4}$	. . . .	hexagonal,
hexagonal . . .	$= \frac{4}{5}$	. . . .	heptagonal,
m gonal	$= \frac{m-2}{m-1}$	. . . .	$(m+1)$gonal.

2. Let us compare the solid angles of the five regular bodies. In these bodies, if m be the number of sides of each face; n the number of planes which meet at each solid angle; $\frac{1}{2}\bigcirc$ = half the circumference or 180°; and A the plane angle made by two adjacent faces: then we have $\sin \frac{1}{2}A = \dfrac{\cos \frac{1}{2n}\bigcirc}{\sin \frac{1}{2m}\bigcirc}$.

This theorem gives, for the plane angle formed by every two contiguous faces of the tetraëdon, 70°31′42″; of the hexaëdron, 90°; of the octaëdron, 109°28′18″; of the dodecaëdron, 116°33′54″; of the icosaëdron, 138°11′23″. But, in these polyedra, the number of faces meeting about each solid angle,

3, 3, 4, 3, 5 respectively. Consequently the several solid angles will be determined by the subjoined proportions:

		Solid Angle.	
$360° : 3.70°31'42'' - 180°$	$:: 1000 :$	$87{\cdot}73611$	Tetraëdron.
$360° : 3.90° - 180°$	$:: 1000 :$	250	Hexaëdron.
$360° : 4.109°28'18'' - 360°$	$:: 1000 :$	$216{\cdot}35185$	Octaëdron.
$360° : 3.116°33'54'' - 180°$	$:: 1000 :$	$471{\cdot}395$	Dodecaëdron.
$360° : 5.138°11'23'' - 540°$	$:: 1000 :$	$419{\cdot}30169$	Icosaëdron.

3. The solid angles at the vertices of cones, will be determined by means of the spheric segments cut off at the bases of those cones; that is, if right cones, instead of having plane bases, had bases formed of the segments of equal spheres, whose centres were the vertices of the cones, the surfaces of those segments would be measures of the solid angles at the respective vertices. Now, the surfaces of spheric segments, are to the surface of the hemisphere, as their altitudes, to the radius of the sphere; and therefore the solid angles at the vertices of right cones, will be to the maximum solid angle, as the excess of the slant side above the axis of the cone, to the slant side of the cone. Thus, if we wish to ascertain the solid angles at the vertices of the equilateral and the right-angled cones; the axis of the former is $\frac{1}{2}\sqrt{3}$, of the latter, $\frac{1}{2}\sqrt{2}$, the slant side of each being unity. Hence,

	Angle at vertex.	
$1 : 1 - \frac{1}{2}\sqrt{3} :: 1000 :$	$133{\cdot}97464$,	equilateral cone,
$1 : 1 - \frac{1}{2}\sqrt{2} :: 1000 :$	$292{\cdot}89322$,	right-angled cone.

4. From what has been said, the mode of determining the solid angles at the vertices of pyramids will be sufficiently obvious. If the pyramids be regular ones, if N be the number of faces meeting about the vertical angle in one, and A the angle of inclination of each two of its plane faces; if n be the number of planes meeting about the vertex of the other, and a the angle of inclination of each two of its faces: then will the vertical angle of the former, be to the vertical angle of the latter pyramid, as $\text{NA} - (\text{N} - 2)\ 180°$, to $na - (n - 2)\ 180°$.

If a cube be cut by diagonal planes, into 6 equal pyramids with square bases, their vertices all meeting at the centre of the circumscribing sphere; then each of the solid angles, made by the four planes meeting at each vertex, will be $\frac{1}{3}$ of the maximum solid angle; and each of the solid angles at the bases of the pyramids, will be $\frac{1}{12}$ of the maximum solid angle. Therefore, each solid angle at the base of such pyramid, is *one-fourth* of the solid angle at its vertex: and, if the angle at the vertex be bisected, as described below, either of the solid angles arising from the bisection, will be double of either solid angle at the base. Hence also, and from the first subdivision

subdivision of this scholium, each solid angle of a prism, with equilateral triangular base, will be *half* each vertical angle of these pyramids, and *double* each solid angle at their bases.

The angles made by one plane with another, must be ascertained, either by measurement or by computation, according to circumstances. But, the general theory being thus explained, and illustrated, the further application of it is left to the skill and ingenuity of geometers; the following simple example, merely, being added here.

Ex. Let the solid angle at the vertex of a square pyramid be bisected.

1st. Let a plane be drawn through the vertex and any two opposite angles of the base, that plane will bisect the solid angle at the vertex; forming two trilateral angles, each equal to half the original quadrilateral angle.

2dly. Bisect either diagonal of the base, and draw *any* plane to pass through the point of bisection and the vertex of the pyramid; such plane, if it do *not* coincide with the former, will divide the quadrilateral solid angle into two equal quadrilateral solid angles. For this plane, produced, will bisect the great circle diagonal of the spherical parallelogram cut off by the base of the pyramid; and any great circle bisecting such diagonal is known to bisect the spherical parallelogram, or square; the plane, therefore, bisects the solid angle.

Cor. Hence an indefinite number of planes may be drawn, each to bisect a given quadrilateral solid angle.

SECTION II.

Resolution of Spherical Triangles.

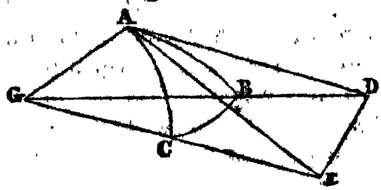
THE different cases of spherical trigonometry, like those in plane trigonometry, may be solved either geometrically or algebraically. We shall here adopt the analytical method, as well on account of its being more compatible with brevity, as because of its correspondence and connexion with the substance of the preceding chapter*. The whole doctrine may be comprehended in the subsequent problems and theorems.

* For the geometrical method, the reader may consult Simson's or Playfair's Euclid, or Bishop Horsley's Elementary Treatises on Practical Mathematics.

PROBLEM I.

To Find Equations, from which may be deduced the Solution of all the Cases of Spherical Triangles.

Let ABC be a spherical triangle; AD the tangent, and GD the secant, of the arc AB; AE the tangent, and GE the secant, of the arc AC; let the capital letters A, B, C, denote the angles of the triangle, and the small letters a, b, c, the opposite sides BC, AC, AB. Then the first equations in art. 6 Pl. Trig.



applied to the two triangles ADE, GDE, give, for the former, $DE^2 = \tan^2 b + \tan^2 c - \tan b . \tan c . \cos A$; for the latter, $DE^2 = \sec^2 b + \sec^2 c - \sec b . \sec c . \cos a$. Subtracting the first of these equations from the second, and observing that $\sec^2 b - \tan^2 b = R^2 = 1$, we shall have, after a little reduction, $1 + \frac{\sin b . \sin c}{\cos b . \cos c} \cos A - \frac{\cos a}{\cos b . \cos c} = 0$. Whence the three following symmetrical equations are obtained:

$$\left.\begin{aligned} \cos a &= \cos b . \cos c + \sin b . \sin c . \cos A \\ \cos b &= \cos a . \cos c + \sin a . \sin c . \cos B \\ \cos c &= \cos a . \cos b + \sin a . \sin b . \cos C \end{aligned}\right\} \text{(I.)}$$

THEOREM VII.

In Every Spherical Triangle, the Sines of the Angles are Proportional to the Sines of their Opposite Sides.

If, from the first of the equations marked I, the value of cos A be drawn, and substituted for it in the equation $\sin^2 A = 1 - \cos^2 A$, we shall have

$$\sin^2 A = 1 - \frac{\cos^2 a + \cos^2 b . \cos^2 c - 2 \cos a . \cos b . \cos c}{\sin^2 b . \sin^2 c}.$$

Reducing the terms of the second side of this equation to a common denominator, multiplying both numerator and denominator by $\sin^2 a$, and extracting the sq. root, there will result

$$\sin A = \sin a . \frac{\sqrt{(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a . \cos b . \cos c)}}{\sin a . \sin b . \sin c}.$$

Here, if the whole fraction which multiplies sin a, be denoted by K (see art. 8 chap. iii), we may write $\sin A = K . \sin a$. And, since the fractional factor, in the above equation, contains terms in which the sides a, b, c, are alike affected, we have

have similar equations for sin B, and sin C. That is to say, we have.
sin A = K . sin a . . . sin B = K . sin b . . . sin C = K . sin c.
Consequently, $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$ . . . (II.) which is the algebraical expression of the theorem.

THEOREM VIII.

In Every Right-Angled Spherical Triangle, the Cosine of the Hypothenuse, is equal to the Product of the Cosines of the Sides Including the right angle.

For, if A be measured by $\frac{1}{4}$O, its cosine becomes nothing, and the first of the equations I becomes cos a = cos b . cos c. Q. E. D.

THEOREM IX.

In Every Right-Angled Spherical Triangle, the Cosine of either Oblique Angle, is equal to the Quotient of the Tangent of the Adjacent Side divided by the Tangent of the Hypothenuse.

If, in the second of the equations I, the preceding value of cos a be substituted for it, and for sin a its value tan a . cos a = cos a . cos b . cos c; then, recollecting that $1 - \cos^2 c = \sin^2 c$, there will result, tan a . cos c . cos B = sin c: whence it follows that,

$$\tan a \cdot \cos B = \tan c, \text{ or } \cos B = \frac{\tan c}{\tan a}.$$

$$\text{Thus also it is found that } \cos C = \frac{\tan b}{\tan a}.$$

THEOREM X.

In Any Right-Angled Spherical Triangle, the Cosine of one of the Sides about the right angle, is equal to the Quotient of the Cosine of the Opposite angle divided by the Sine of the Adjacent angle.

From th. 7, we have $\frac{\sin B}{\sin A} = \frac{\sin b}{\sin a}$; which, when A is a right angle, becomes simply $\sin B = \frac{\sin b}{\sin a}$. Again, from th. 9, we have $\cos C = \frac{\tan b}{\tan a}$. Hence, by division,

$$\frac{\cos C}{\sin B} = \frac{\tan b}{\sin b} \cdot \frac{\sin a}{\tan a} = \frac{\cos a}{\cos b}.$$

Now, th. 8 gives $\frac{\cos a}{\cos b} = \cos c$. Therefore $\frac{\cos C}{\sin B} = \cos c$; and in like manner, $\frac{\cos B}{\sin C} = \cos b$. Q. E. D.

THEOREM

THEOREM XI.

In Every Right-Angled Spherical Triangle, the Tangent of either of the Oblique Angles, is equal to the Quotient of the Tangent of the Opposite Side, divided by the Sine of the other Side about the right angle.

For, since $\sin B = \frac{\sin b}{\sin a}$, and $\cos B = \frac{\tan c}{\tan a}$,

we have $\frac{\sin B}{\cos B} = \frac{\sin b}{\sin a} \cdot \frac{\tan a}{\tan c}$.

Whence, because (th. 8) $\cos a = \cos b \cdot \cos c$, and since $\sin a = \cos a \cdot \tan a$, we have

$$\tan B = \frac{\sin b}{\cos a \cdot \tan c} = \frac{\sin b}{\cos b \cdot \cos c \cdot \tan c} = \frac{\sin b}{\cos b} \cdot \frac{1}{\cos c \cdot \tan c} = \frac{\tan b}{\sin c}.$$

In like manner, $\tan C = \frac{\tan c}{\sin b}$. Q. E. D.

THEOREM XII.

In Every Right-Angled Spherical Triangle, the Cosine of the Hypothenuse, is equal to the Quotient of the Cotangent of one of the Oblique Angles, divided by the Tangent of the Other Angle.

For, multiplying together the resulting equations of the preceding theorem, we have

$$\tan B \cdot \tan C = \frac{\tan b}{\sin b} \cdot \frac{\tan c}{\sin c} = \frac{1}{\cos b \cdot \cos c}.$$

But, by th. 8, $\cos b \cdot \cos c = \cos a$.

Therefore $\tan B \cdot \tan C = \frac{1}{\cos a}$, or $\cos a = \frac{\cot C}{\tan B}$. Q. E. D.

THEOREM XIII.

In Every Right-Angled Spherical Triangle, the Sine of the Difference between the Hypothenuse and Base, is equal to the Continued Product of the Sine of the Perpendicular, Cosine of the Base, and Tangent of Half the Angle Opposite to the Perpendicular; or equal to the Continued Product of the Tangent of the Perpendicular, Cosine of the Hypothenuse, and Tangent of Half the Angle Opposite to the Perpendicular *.

* This theorem is due to M. Prony, who published it without demonstration in the *Connaissance des Temps* for the year 1808, and made use of it in the construction of a chart of the course of the Po.

Here,

Here, retaining the same notation, since we have $\sin a = \frac{\sin b}{\sin B}$, and $\cos B = \frac{\tan c}{\tan a}$; if for the tangents there be substituted their values in sines and cosines, there will arise,

$$\sin c \cdot \cos a = \cos B \cdot \cos c \cdot \sin a = \cos B \cdot \cos c \cdot \frac{\sin b}{\sin B}.$$

Then substituting for $\sin a$, and $\sin c \cdot \cos a$, their values in the known formula (equ. v chap. iii) viz,

in $\sin (a - c) = \sin a \cdot \cos c - \cos a \cdot \sin c$,

and recollecting that $\frac{1 - \cos B}{\sin B} = \tan \frac{1}{2}B$,

it will become, $\sin (a - c) = \sin b \cdot \cos c \cdot \tan \frac{1}{2}B$;

which is the first part of the theorem: and, if in this result we introduce, instead of $\cos c$, its value $\frac{\cos a}{\cos b}$ (th. 8), it will be transformed into $\sin (a-c) = \tan b \cdot \cos a \cdot \tan \frac{1}{2}B$; which is the second part of the theorem. Q. E. D.

Cor. This theorem leads manifestly to an analagous one with regard to rectilinear triangles, which, if h, b, and p denote the hypothenuse, base, and perpendicular, and B, P, the angles respectively opposite to b, p; may be expressed thus:

$$h - b = p \cdot \tan \tfrac{1}{2}P \;.\;.\;.\;.\;.\; h - p = b \cdot \tan \tfrac{1}{2}B.$$

These theorems may be found useful in reducing inclined lines to the plane of the horizon.

PROBLEM II.

Given the Three Sides of a Spherical Triangle; it is required to find Expressions for the Determination of the Angles.

Retaining the notation of prob. 1, in all its generality, we soon deduce from the equations marked I in that problem, the following; viz,

$$\left.\begin{aligned} \cos A &= \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c} \\ \cos B &= \frac{\cos b - \cos a \cdot \cos c}{\sin a \cdot \sin c} \\ \cos C &= \frac{\cos c - \cos a \cdot \cos b}{\sin a \cdot \sin b} \end{aligned}\right\}$$

As these equations, however, are not well suited for logarithmic computation; they must be so transformed, that their second members will resolve into factors. In order to this, substitute in the known equation $1 - \cos A = 2 \sin^2 \frac{1}{2}A$, the preceding value of $\cos A$, and there will result

$$2 \sin^2 \tfrac{1}{2}A = \frac{\cos (b - c) - \cos a}{\sin b \cdot \sin c}.$$

But,

But, because $\cos B' - \cos A' = 2 \sin \frac{1}{2}(A' + B') . \sin \frac{1}{2}(A' - B')$ (art. 25 ch. iii), and consequently,

$$\cos (b - c) - \cos a = 2 \sin \frac{a + b - c}{2} . \sin \frac{a + c - b}{2}:$$

we have, obviously,

$$\sin^2 \tfrac{1}{2}A = \frac{\sin \frac{1}{2}(a + b - c) . \sin \frac{1}{2}(a + c - b)}{\sin b . \sin c}$$

Whence, making $s = a + b + c$, there results

$$\left.\begin{array}{ll} & \sin \tfrac{1}{2}A = \sqrt{\dfrac{\sin (\frac{1}{2}s - b) . \sin (\frac{1}{2}s - c)}{\sin b . \sin c}}. \\ \text{So, also,} & \sin \tfrac{1}{2}B = \sqrt{\dfrac{\sin (\frac{1}{2}s - a) . \sin (\frac{1}{2}s - c)}{\sin a . \sin c}}. \\ \text{And,} & \sin \tfrac{1}{2}C = \sqrt{\dfrac{\sin (\frac{1}{2}s - a) . \sin (\frac{1}{2}s - b)}{\sin a . \sin b}}. \end{array}\right\} \text{(III.)}$$

The expressions for the tangents of the half angles, might have been deduced with equal facility: and we should have obtained, for example,

$$\tan \tfrac{1}{2}A = \sqrt{\frac{\sin (\frac{1}{2}s - b) . \sin (\frac{1}{2}s - c)}{\sin \frac{1}{2}s . \sin \frac{1}{2}(s - a)}} \quad (iii.)$$

Thus again, the expressions for the cosine and cotangent of half one of the angles, are

$$\cos \tfrac{1}{2}A = \sqrt{\frac{\sin \frac{1}{2}s . \sin \frac{1}{2}(s - a)}{\sin b . \sin c}}.$$

$$\cot \tfrac{1}{2}A = \sqrt{\frac{\sin \frac{1}{2}s . \sin \frac{1}{2}(s - a)}{\sin (\frac{1}{2}s - b) . \sin (\frac{1}{2}s - c)}}.$$

The three latter flowing naturally from the former, by means of the values $\tan = \frac{\sin}{\cos}$, $\cot = \frac{\cos}{\sin}$. (art 4 ch. iii.)

Cor. 1. When two of the sides, as b and c, become equal, then the expression for $\sin \frac{1}{2}A$ becomes

$$\sin \tfrac{1}{2}A = \frac{\sin (\frac{1}{2}s - b)}{\sin b} = \frac{\sin \frac{1}{2}a}{\sin b}.$$

Cor. 2. When all the three sides are equal, or $a = b = c$, then $\sin \frac{1}{2}A = \frac{\sin \frac{1}{2}a}{\sin a}$.

Cor. 3. In this case, if $a = b = c = 90°$; then $\sin \frac{1}{2}A = \frac{\frac{1}{2}\sqrt{2}}{1} = \frac{1}{2}\sqrt{2} = \sin 45°$: and $A = B = C = 90°$.

Cor. 4. If $a = b = c = 60°$: then $\sin \frac{1}{2}A = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{3}\sqrt{3} = \sin 35°15'51''$: and $A = B = C = 70°31'42''$, the same as the angle between two contiguous planes of a tetraedron.

Cor. 5. If $a = b = c$ were assumed $= 120°$: then $\sin \frac{1}{2}A = \frac{\sin 60°}{\sin 120°} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}\sqrt{3}} = 1$; and $A = B = C = 180°$; which shows that no such triangle can be constructed (conformably to th. 2); but that the three sides would, in such case, form three continued arcs completing a great circle of the sphere.

PROBLEM

PROBLEM III.

Given the Three Angles of a Spherical Triangle, to find Expressions for the Sides.

If from the first and third of the equations marked I (prob. 1), cos c be exterminated, there will result,

$$\cos A \,.\, \sin c + \cos C \,.\, \sin a \,.\, \cos b = \cos a \,.\, \sin b.$$

But, it follows from th. 7, that $\sin c = \frac{\sin a \;\; \sin C}{\sin A}$. Substituting for sin c this value of it, and for $\frac{\cos A}{\sin A}$, $\frac{\cos a}{\sin a}$, their equivalents cot A, cot a, we shall have,

$$\cot A \,.\, \sin C + \cos C \,.\, \cos b = \cot a \,.\, \sin b.$$

Now, $\cot a \,.\, \sin b = \frac{\cos a}{\sin a} \,.\, \sin b = \cos a \,.\, \frac{\sin b}{\sin a} = \cos a \,.\, \frac{\sin B}{\sin A}$, (th. 7). So that the preceding equation at length becomes,

$$\cos A \,.\, \sin C = \cos a \;\; \sin B - \sin A \,.\, \cos C \,.\, \cos b.$$

In like manner, we have,

$$\cos B \,.\, \sin C = \cos b \,.\, \sin A - \sin B \,.\, \cos C \,.\, \cos a.$$

Exterminating cos b from these, there results

$$\left.\begin{array}{l} \cos A = \cos a \,.\, \sin B \,.\, \sin C - \cos B \,.\, \cos C. \\ \text{So likewise} \left\{ \begin{array}{l} \cos B = \cos b \,.\, \sin A \,.\, \sin C - \cos A \,.\, \cos C. \\ \cos C = \cos c \,.\, \sin A \,.\, \sin B - \cos A \,.\, \cos B. \end{array}\right. \end{array}\right\} \text{(IV.)}$$

This system of equations is manifestly analogous to equation I; and if they be reduced in the manner adopted in the last problem, they will give

$$\left.\begin{array}{l} \sin \tfrac{1}{2}a = \sqrt{-\frac{\cos \frac{1}{2}(A+B+C) \,.\, \cos \frac{1}{2}(B+C-A)}{\sin B \;\; \sin C}}. \\ \sin \tfrac{1}{2}b = \sqrt{-\frac{\cos \frac{1}{2}(A+B+C) \;\; \cos \frac{1}{2}(A+C-B)}{\sin A \;\; \sin C}}. \\ \sin \tfrac{1}{2}c = \sqrt{-\frac{\cos \frac{1}{2}(A+B+C) \,.\, \cos \frac{1}{2}(A+B-C)}{\sin A \,.\, \sin B}}. \end{array}\right\} \text{(V.)}$$

The expression for the tangent of half a side is

$$\tan \tfrac{1}{2}a = \sqrt{-\frac{\cos \frac{1}{2}(A+B+C) \,.\, \cos \frac{1}{2}(B+C-A)}{\cos \frac{1}{2}(A+C-B) \,.\, \cos \frac{1}{2}(A+B-C)}}.$$

The values of the cosines and cotangents are omitted, to save room; but are easily deduced by the student.

Cor. 1. When two of the angles, as B and C, become equal, then the value of cos $\frac{1}{2}a$ becomes $\cos \frac{1}{2}a = \frac{\cos \frac{1}{2}A}{\sin B}$.

Cor. 2. When A = B = C; then $\cos \frac{1}{2}a = \frac{\cos \frac{1}{2}A}{\sin A}$.

Cor. 3. When A = B = C = 90°, then $a = b = c = 90°$.

Cor. 4. If A = B = C = 60°; then $\cos \frac{1}{2}a = \frac{\sin 60°}{\sin 60} = 1$. So that $a = b = c = 0$. Consequently no such triangle can be constructed: conformably to th. 3.

Cor. 5. If $A = B = C = 120°$: then $\cos \frac{1}{2}a = \frac{\cos 60°}{\sin 120°} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{3}\sqrt{3} = \cos 54°44'9''$. Hence $a = b = c = 109°28'18''$.

Schol. If, in the preceding values of $\sin \frac{1}{2}a$, $\sin \frac{1}{2}b$, &c, the quantities under the radical were negative in reality, as they are in appearance, it would obviously be impossible to determine the value of $\sin \frac{1}{2}a$, &c. But this value is in fact always real. For, in general, $\sin (x - \frac{1}{4}\bigcirc) = -\cos x$: therefore, $\sin (\frac{A+B+C}{2} - \frac{1}{4}\bigcirc) = -\cos \frac{1}{2}(A + B + C)$; a quantity which is always positive, because, as $A + B + C$ is necessarily comprised between $\frac{1}{2}\bigcirc$ and $\frac{3}{2}\bigcirc$, we have $\frac{1}{2}(A + B + C) - \frac{1}{4}\bigcirc$ greater than nothing, and less than $\frac{1}{2}\bigcirc$. Further, any one side of a spherical triangle being smaller than the sum of the other two, we have, by the property of the polar triangle (theorem 4), $\frac{1}{2}\bigcirc - A$ less than $\frac{1}{2}\bigcirc - B + \frac{1}{2}\bigcirc - C$; whence $\frac{1}{2}(B + C - A)$ is less than $\frac{1}{4}\bigcirc$; and of course its cosine is positive.

PROBLEM IV.

Given Two Sides of a Spherical Triangle, and the Included Angle; to obtain Expressions for the Other Angles.

1. In the investigation of the last problem, we had

$$\cos A \,.\, \sin c = \cos a \,.\, \sin b - \cos C \,.\, \sin a \,.\, \cos b:$$

and by a simple permutation of letters, we have

$$\cos B \,.\, \sin c = \cos b \,.\, \sin a - \cos C \,.\, \sin b \,.\, \cos a:$$

adding together these two equations, and reducing, we have

$$\sin c\,(\cos A + \cos B) = (1 - \cos C)\sin (a + b).$$

Now, we have from theor. 7,

$$\frac{\sin a}{\sin A} = \frac{\sin c}{\sin C}, \text{ and } \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Freeing these equations from their denominators, and respectively adding and subtracting them, there results

$$\sin c\,(\sin A + \sin B) = \sin C\,(\sin a + \sin b),$$

$$\text{and } \sin c\,(\sin A - \sin B) = \sin C\,(\sin a - \sin b).$$

Dividing each of these two equations by the preceding, there will be obtained

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin C}{1 - \cos C} \,.\, \frac{\sin a + \sin b}{\sin (a + b)},$$

$$\frac{\sin A - \sin B}{\cos A + \cos B} = \frac{\sin C}{1 - \cos C} \,.\, \frac{\sin a - \sin b}{\sin (a + b)}.$$

Comparing these with the equations in arts. 25, 26, 27, ch. iii, there will at length result

$$\left.\begin{aligned} \tan \tfrac{1}{2}(A + B) &= \cot \tfrac{1}{2}C \,.\, \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \\ \tan \tfrac{1}{2}(A - B) &= \cot \tfrac{1}{2}C \,.\, \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \end{aligned}\right\} \;..\; \text{(VI.)}$$

Cor.

Cor. When $a = b$, the first of the above equations becomes $\tan A = \tan B = \cot \frac{1}{2}C \cdot \sec a$.

And in this case it will be, as rad : $\sin \frac{1}{2}C$ :: $\sin a$ or $\sin b$: $\sin \frac{1}{2}c$.

And, as rad : $\cos A$ or $\cos B$ :: $\tan a$ or $\tan b$: $\tan \frac{1}{2}c$.

2. The preceding values of $\tan \frac{1}{2}(A + B)$, $\tan \frac{1}{2}(A - B)$ are very well fitted for logarithmic computation: it may, notwithstanding, be proper to investigate a theorem which will at once lead to one of the angles, by means of a subsidiary angle. In order to this, we deduce immediately from the second equation in the investigation of prob. 3,

$$\cot A = \frac{\cot a \cdot \sin b}{\sin C} - \cot C \cdot \cos b.$$

Then, choosing the subsidiary angle ϕ so that

$$\tan \phi = \tan a \cdot \cos C,$$

that is, finding the angle ϕ, whose tangent is equal to the product $\tan a \cdot \cos C$, which is equivalent to dividing the original triangle into two right-angled triangles, the preceding equation will become

$$\cot A = \cot C(\cot \phi \cdot \sin b - \cos b) = \frac{\cot C}{\sin \phi}(\cos \phi \cdot \sin b - \sin \phi \cdot \cos b).$$

And this, since $\sin(b - \phi) = \cos \phi \cdot \sin b - \sin \phi \cdot \cos b$, becomes

$$\cot A = \frac{\cot C}{\sin \phi} \cdot \sin(b - \phi).$$

Which is a very simple and convenient expression.

PROBLEM V.

Given Two Angles of a Spherical Triangle, and the Side Comprehended between them; to find Expressions for the Other Two Sides.

1. Here, a similar analysis to that employed in the preceding problem, being pursued with respect to the equations IV, in prob. 3, will produce the following formulæ:

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \frac{\sin c}{1 + \cos c} \cdot \frac{\sin A + \sin B}{\sin (A + B)},$$

$$\frac{\sin a - \sin b}{\cos a + \cos b} = \frac{\sin c}{1 + \cos c} \cdot \frac{\sin A - \sin B}{\sin (A + B)}.$$

Whence, as in prob. 4, we obtain

$$\left.\begin{aligned} \tan \tfrac{1}{2}(a + b) &= \tan \tfrac{1}{2}c \cdot \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)}, \\ \tan \tfrac{1}{2}(a - b) &= \tan \tfrac{1}{2}c \cdot \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)}. \end{aligned}\right\} \text{(VII*.)}$$

* The formulæ marked VI, and VII, converted into analogies, by making the denominator of the second member the first term, the other two factors the second and third terms, and the first member of the equation, the fourth term of the proportion, as

2. If

2. If it be wished to obtain a side at once, by means of a subsidiary angle; then, find ϕ so that $\frac{\cot A}{\cos c} = \tan \phi$; then will $\cot a = \frac{\cot c}{\cos \phi} \cdot \cos (B - \phi)$.

PROBLEM VI.

Given Two Sides of a Spherical Triangle, and an Angle Opposite to one of them; to find the Other Opposite Angle.

Suppose the sides given are a, b, and the given angle B: then from theor. 7, we have $\sin A = \frac{\sin a \cdot \sin B}{\sin b}$; or, $\sin A$, a fourth proportional to $\sin b$, $\sin B$, and $\sin a$.

PROBLEM VII.

Given Two Angles of a Spherical Triangle, and a Side Opposite to one of them; to find the Side Opposite to the other.

Suppose the given angles are A, and B, and b the given side: then th. 7, gives $\sin a = \frac{\sin b \cdot \sin A}{\sin B}$; or, $\sin a$, a fourth proportional to $\sin B$, $\sin b$, and $\sin A$.

Scholium.

In problems 2 and 3, if the circumstances of the question leave any doubt, whether the arcs or the angles sought, are greater or less than a quadrant, or than a right angle, the difficulty will be entirely removed by means of the table of mutations of signs of trigonometrical quantities, in different quadrants, marked VII in chap 3. In the 6th and 7th problems, the question proposed will often be susceptible of two solutions: by means of the subjoined table the student may always tell when this will or will not be the case.

1. With the data a, b, and B, there can be only one solution when $B = \frac{1}{4}○$ (a right angle),

or, when $B < \frac{1}{4}○ \ldots\ldots a < \frac{1}{4}○ \ldots\ldots b > \frac{1}{2}a$

$B < \frac{1}{4}○ \ldots\ldots a > \frac{1}{4}○ \ldots\ldots b > ○ - a$,

$B > \frac{1}{4}○ \ldots\ldots a < \frac{1}{4}○ \ldots\ldots b < ○ - a$,

$B > \frac{1}{4}○ \ldots\ldots a > \frac{1}{4}○ \ldots\ldots b < a$.

$\cos \frac{1}{2}(a + b) : \cos \frac{1}{2}(a - b) :: \cot \frac{1}{2}C : \tan \frac{1}{2}(A + B)$,
$\sin \frac{1}{2}(a + b) : \sin \frac{1}{2}(a - b) :: \cot \frac{1}{2}C : \tan \frac{1}{2}(A - B)$, &c. &c.
are called *the Analogies of Napier*, being invented by that celebrated geometer. He likewise invented other rules for spherical trigonometry, known by the name of *Napier's Rules for the circular parts*; but these, notwithstanding their ingenuity, are not inserted here; because they are too artificial to be applied by a young computist, to every case that may occur, without considerable danger of misapprehension and error.

The

The triangle is susceptible of two forms and solutions

when $B < \frac{1}{4}\bigcirc \ldots\ldots a < \frac{1}{4}\bigcirc \ldots\ldots b < a$,

$B < \frac{1}{4}\bigcirc \ldots\ldots a > \frac{1}{4}\bigcirc \ldots\ldots b < \frac{1}{2}\bigcirc - a$,

$B > \frac{1}{4}\bigcirc \ldots\ldots a < \frac{1}{4}\bigcirc \ldots\ldots b > \frac{1}{2}\bigcirc - a$,

$B > \frac{1}{4}\bigcirc \ldots\ldots a > \frac{1}{4}\bigcirc \ldots\ldots b > a$,

B < or > $\frac{1}{4}\bigcirc \ldots\ldots a = \frac{1}{4}\bigcirc$.

2. With the data A, B, and b, the triangle can exist but in one form,

when $b = \frac{1}{4}\bigcirc$ (one quadrant),

$b > \frac{1}{4}\bigcirc \ldots\ldots A > \frac{1}{4}\bigcirc \ldots\ldots B < A$,

$b > \frac{1}{4}\bigcirc \ldots\ldots A < \frac{1}{4}\bigcirc \ldots\ldots B < \frac{1}{2}\bigcirc - A$,

$b < \frac{1}{4}\bigcirc \ldots\ldots A > \frac{1}{4}\bigcirc \ldots\ldots B > \frac{1}{2}\bigcirc - A$,

$b < \frac{1}{4}\bigcirc \ldots\ldots A < \frac{1}{4}\bigcirc \ldots\ldots B > A$.

It is susceptible of two forms,

when $b > \frac{1}{4}\bigcirc \ldots\ldots A > \frac{1}{4}\bigcirc \ldots\ldots B > A$,

$b > \frac{1}{4}\bigcirc \ldots\ldots A < \frac{1}{4}\bigcirc \ldots\ldots B > \frac{1}{2}\bigcirc - A$,

$b < \frac{1}{4}\bigcirc \ldots\ldots A > \frac{1}{4}\bigcirc \ldots\ldots B < \frac{1}{2}\bigcirc - A$,

$b < \frac{1}{4}\bigcirc \ldots\ldots A < \frac{1}{4}\bigcirc \ldots\ldots B < A$,

b < or > $\frac{1}{4}\bigcirc \ldots\ldots A = \frac{1}{4}\bigcirc$.

It may here be observed, that all the analogies and formulæ, of spherical trigonometry, in which *cosines* or *cotangents* are not concerned, may be applied to *plane* trigonometry; taking care to use only a *side* instead of the *sine* or the *tangent of a side*; or the *sum* or *difference* of the sides instead of the *sine* or *tangent* of such sum or difference. The reason of this is obvious: for analogies or theorems raised, not only from the consideration of a triangular figure, but the curvature of the sides also, are of consequence more general; and therefore, though the curvature should be deemed evanescent, by reason of a diminution of the surface, yet what depends on the *triangle* alone will remain, notwithstanding.

We have now deduced all the rules that are essential in the operations of spherical trigonometry; and explained under what limitations ambiguities may exist. That the student, however, may want nothing further to direct his practice in this branch of science, we shall add three tables, in which the several formulæ, already given, are respectively applied to the solution of all the cases of right and oblique-angled spherical triangles, that can possibly occur.

TABLE I.

For the Solution of all the cases of Right-Angled Spherical Triangles.

Given.	Required.	Values of the terms required.	Cases in which the terms required are less than 90°.
I. Hypothenuse, and one leg.	Angle opposite to the given leg.	Its sin $= \frac{\text{sin given leg}}{\text{sin hypoth.}}$.	If the given leg be less than 90°.
	Angle adjacent to the given leg.	Its cos $= \frac{\text{tan given leg}}{\text{tan hypoth.}}$.	If the things given be of the same affection.
	Other leg.	Its cos $= \frac{\text{cos hypoth.}}{\text{cos given leg}}$.	Idem.
II. One leg, and its opposite angle.	Hypothenuse.	Its sin $= \frac{\text{sin given leg}}{\text{sin given ang}}$.	Ambiguous.
	Other leg.	Its sin $= \frac{\text{tan given leg}}{\text{tan given ang.}}$.	Idem.
	Other angle.	Its sin $= \frac{\text{cos given ang.}}{\text{cos given leg}}$.	Idem.
III. One leg, and the adjacent angle.	Hypothenuse.	Its tan $= \frac{\text{tan given leg}}{\text{cos given ang.}}$.	If the things given be of like affection.
	Other angle.	Its cos $=$ cos giv. leg $\times$ sin giv. ang.	If the given leg be less than 90°.
	Other leg.	Its tan $=$ sin giv. leg. tan giv. ang.	If the given angle be less than 90°.

IV. Hypothenuse, and one angle.	Adjacent leg.	Its tan = tan hyp × cos giv. ang.	If the things given be of like affection.
	Leg opp. to the given angle.	Its sin = sin hyp × sin giv. ang.	If the given angle be acute.
	Other angle.	Its tan = $\frac{\text{cot giv. angle}}{\text{cos hypothen.}}$	If the things given be of like affection.
V. The two legs.	Hypothenuse.	Its cos = rectan. cos giv. legs.	If the given legs be of like affection.
	Either of the angles.	Its tan = $\frac{\text{tan oppos. leg}}{\text{sin adjac. leg}}$.	If the opposite leg be less than 90°.
VI. The two angles.	Hypothenuse.	Its cos = rect. cot giv. angles.	If the angles be of like affection.
	Either of the legs.	Its cos = $\frac{\text{cos opposite angle}}{\text{sin adjacent angle}}$.	If the opposite angle be acute.

In working by the logarithms, the student must observe that when the resulting logarithm is the log. of a quotient, 10 must be *added* to the index; when it is the log. of a product, 10 must be *subtracted* from the index. Thus when the two angles are given,

Log. cos hypothen. = log. cos one angle + log. cos other angle − 10:
Log. cos either leg = log. cos opp. angle − log. sin adjac. angle + 10.

In a quadrantal triangle, if the quadrantal side be called radius, the supplement of the angle opposite to that side be called hypothenuse, the other sides be called angles, and their opposite angles be called legs: then the solutions of all the cases will be as in this table; merely changing *like* for *unlike* in the determinations.

TABLE II.—*For the Solution of Oblique-Angled Spherical Triangles.*

An angle or a side being divided by a perpendicular, the first and second segments are denoted by 1 seg. and 2 seg.

Given.	Required.		Values of the Quantities required.
I. Two angles and a side opposite to one of them.	The side opp. to other angle.	By the common analogy.	Sines of angles are as sines of oppos. sides.
	Third side.	Let fall a per. on the side contained between the given angles.	Tan 1 seg. of this side = cos adj. angle × tan given side.
			Sin 2 seg. = $\frac{\text{sin 1 seg.} \times \text{tan ang. adj. given side}}{\text{tan ang. opp. given side}}$.
	Third angle.	Let fall a per. as before.	Cot 1 seg. of this ang. = cos giv. side × tan adj. angle.
			Sin 2 seg. = $\frac{\text{sin 1 seg.} \times \text{cos ang. opp. given side}}{\text{cos ang. adj. given side}}$.
II. Two sides and an angle opposite to one of them.	The angle opp. to the other side.	By the common analogy.	Sines of sides are as sines of their opposite angles.
	Angle included between the given sides.	Let fall a perpendicular from the included angle.	Cot 1 seg. ang. req. = tan given ang. × cos adj. side.
			Cos 2 seg = $\frac{\text{cos 1 seg.} \times \text{tan giv. side adj. giv. angle}}{\text{tan side opp. given angle}}$.
	Third side.	Let fall a perpendicular as before.	Tan 1 seg. side req. = cos given ang. × tan adj. side.
			Cos 2 seg. = $\frac{\text{cos 1 seg.} \times \text{cos side opp. given angle}}{\text{cos side adj. given angle}}$.

III. Two sides and the included angle.	An angle oppos. to one of the given sides.	Let fall a perpen. from the third angle.	Tan 1 seg. of div. side = cos giv. ang. × tan side opp. ang. sought. Tan ang. sought $= \frac{\text{tan giv. ang.} \times \text{sin 1 seg.}}{\text{sin 2 seg. of div. side}}$.
	Third side.	Let fall a perpen. on one of the giv. sides.	Tan 1 seg. of div. side = cos giv. ang. × tan other given side. Cos side sought $= \frac{\text{cos side not div.} \times \text{cos 2 seg.}}{\text{cos 1 seg. of side divided}}$.
IV. A side and the two adjacent angles.	A side opposite to one of the given angles.	Let fall a perpendicular on the third side.	Cot 1 seg. of div. ang. = cos giv. side × tan ang. opp. side sought. Tan side sought $= \frac{\text{tan giv. side} \times \text{cos 1 seg. div. ang.}}{\text{cos 2 seg. of divided angle}}$.
	Third angle.	Let fall a perpen. from one of the giv. angles.	Cot 1 seg. div. ang. = cos giv. side × tan other giv. angle. Cos angle sought $= \frac{\text{cos ang. not div.} \times \text{sin 2 seg.}}{\text{sin 1 seg. div. angle}}$.
V. The three sides.	An angle by the sine or cosine of its half.	Let a, b, c, be the sides; A, B, C, the angles, b and c including the angle sought, and $s = a + b + c$. Then, $\sin \frac{1}{2}A = \sqrt{\frac{\sin(\frac{1}{2}s - b) \cdot \sin(\frac{1}{2}s - c)}{\sin b \cdot \sin c}}$ . . . . . . $\cos \frac{1}{2}A = \sqrt{\frac{\sin \frac{1}{2}s \cdot \sin(\frac{1}{2}s - a)}{\sin b \cdot \sin c}}$.	
VI. The three angles.	A side by the sine or cosine of its half.	Let s be the sum of the angles A, B, and C; and let B and C be adjacent to a the side required. Then, $\sin \frac{1}{2}a = \sqrt{- \frac{\cos \frac{1}{2}s \cdot \cos(\frac{1}{2}s - A)}{\sin B \cdot \sin C}}$ . . . . $\cos \frac{1}{2}a = \sqrt{- \frac{\sin(\frac{1}{2}s - B) \cdot \sin(\frac{1}{2}s - C)}{\sin B \cdot \sin C}}$.	

TABLE III.

For the Solution of all the cases of Oblique-Angled Spherical Triangles, by the Analogies of Napier.

Given.	Required.	Values of the Terms required.
I. Two angles and one of their opposite sides.	Side opp. to the other given angle.	By the common analogy, sines of angles as sines of opp. sides.
	Third side.	Tan of its half $= \dfrac{\tan \frac{1}{2} \text{ diff. giv. sides} \times \sin \frac{1}{2} \text{ sum opp. angles}}{\sin \frac{1}{2} \text{ diff. of those angles}}$, $= \dfrac{\tan \frac{1}{2} \text{ sum giv. sides} \times \cos \frac{1}{2} \text{ sum opp. angles}}{\cos \frac{1}{2} \text{ diff. of those angles}}$.
	Third angle.	By the common analogy.
II. Two sides, and an opposite angle.	Angle opposite to the other known side.	By the common analogy.
	Third angle.	Cot of its half $= \dfrac{\tan \frac{1}{2} \text{ diff. other two ang.} \times \sin \frac{1}{2} \text{ sum giv. sides}}{\sin \frac{1}{2} \text{ diff. those sides}}$, $= \dfrac{\tan \frac{1}{2} \text{ sum of other two ang.} \times \cos \frac{1}{2} \text{ sum giv. sides}}{\cos \frac{1}{2} \text{ diff. of those sides}}$.
	Third side.	By the common analogy.

III. Two sides, and the included angle.	The other two angles.	Tan $\frac{1}{2}$ their diff. $= \frac{\text{cot } \frac{1}{2} \text{ giv. ang.} \times \text{sin } \frac{1}{2} \text{ diff. giv. sides}}{\text{sin } \frac{1}{2} \text{ sum of those sides}}$. Tan $\frac{1}{2}$ their sum $= \frac{\text{cot } \frac{1}{2} \text{ giv. ang.} \times \text{cos } \frac{1}{2} \text{ diff. giv. sides}}{\text{cos } \frac{1}{2} \text{ sum of those sides}}$.
	Third side.	By the common analogy.
IV. Two angles, and the side between them.	The other two sides.	Tan $\frac{1}{2}$ their diff. $= \frac{\text{tan } \frac{1}{2} \text{ giv. side} \times \text{sin } \frac{1}{2} \text{ diff. giv. angles}}{\text{sin } \frac{1}{2} \text{ sum of those angles}}$. Tan $\frac{1}{2}$ their sum $= \frac{\text{tan } \frac{1}{2} \text{ giv. side} \times \text{cos } \frac{1}{2} \text{ diff. giv. angles}}{\text{cos } \frac{1}{2} \text{ sum of those angles}}$.
	Third angle.	By the common analogy.
V. The three sides.	Either of the angles.	Let fall a perpen. on the side adjacent to the angle sought. Tan $\frac{1}{2}$ sum or $\frac{1}{2}$ diff. of the seg. of the base $= \frac{\text{tan } \frac{1}{2} \text{ sum} \times \text{tan } \frac{1}{2} \text{ diff. of the sides}}{\text{tan } \frac{1}{2} \text{ base}}$. Cos angle sought = tan adj. seg. × cot adja. side.
VI. The three angles.	Either of the des.	Will be obtained by finding its correspondent angle, in a triangle which has all its parts supplemental to those of the triangle whose three angles are given.

Questions for Exercise in Spherical Trigonometry.

Ex. 1. In the right-angled spherical triangle BAC, right-angled at A, the hypothenuse $a = 78°20'$, and one leg $c = 76°52'$, are given; to find the angles B, and C, and the other leg b.

Here, by table case 1, $\sin C = \frac{\sin c}{\sin a}$;

$\cos B = \frac{\tan c}{\tan a}$; . . . $\cos b = \frac{\cos a}{\cos c}$.

Or, $\log \sin C = \log \sin c - \log \sin a + 10$.
$\log \cos B = \log \tan c - \log \tan a + 10$.
$\log \cos b = \log \cos a - \log \cos c + 10$.

Hence,	$10 + \log \sin c =$	$10 + \log \sin 76°52' =$	19·9884894
	$\log \sin a =$	$\log \sin 78°20' =$	9·9909338
Remains,	$\log \sin C =$	$\log \sin 83°56' =$	9·9975556

Here C is acute, because the given leg is less than 90°.

Again,	$10 + \log \tan c =$	$10 + \log \tan 76°52' =$	20·6320468
	$\log \tan a =$	$\log \tan 78°20' =$	10·6851149
Remains,	$\log \cos B =$	$\log \cos 27°45' =$	9·9169519

B is here acute, because a and c are of like affection.

Lastly,	$10 + \log \cos a =$	$10 + \log \cos 78°20' =$	19·3058189
	$\log \cos c =$	$\log \cos 76°52 =$	9·3564426
Remains,	$\log \cos b =$	$\log \cos 27° 8' =$	9·9493763

where b is less than 90°, because a and c both are so

Ex. 2. In a right-angled spherical triangle, denoted as above, are given $a = 78°20'$, $B = 27°45'$; to find the other sides and angle.

Ans. $b = 27° 8'$, $c = 76°52'$, $C = 83°56'$.

Ex. 3. In a spherical triangle, with A a right angle, given $b = 117°34'$, $c = 31°51'$; to find the other parts

Ans. $a = 113°55'$, $C = 28°51'$, $B = 104° 8'$.

Ex. 4. Given $b = 27°6'$, $c = 76°52'$; to find the other parts. Ans. $a = 78°20'$, $B = 27°45'$, $C = 83°56'$.

Ex. 5. Given $b = 42°12'$, $B = 48°$; to find the other parts.

Ans. $a = 64°40'\frac{1}{2}$, or its supplement,
$c = 54°44'$, or its supplement,
$C = 64°35'$, or its supplement.

Ex. 6. Given $B = 48°$, $C = 64°35'$; required the other parts? Ans. $b = 42°12'$, $c = 54°44'$, $a = 64°40'\frac{1}{2}$.

Ex.

Ex. 7. In the quadrantal triangle ABC, given the quadrantal side $a = 90$, an adjacent angle $C = 42°12'$; and the opposite angle $A = 64°40'$; required the other parts of the triangle?

Ex. 8. In an oblique-angled spherical triangle are given the three sides, viz, $a = 56°40'$, $b = 83°13'$, $c = 114°30'$; to find the angles.

Here, by the fifth case of table 2, we have

$$\sin \tfrac{1}{2}A = \sqrt{\frac{\sin(\frac{1}{2}s - b)\ \sin(\frac{1}{2}s - c)}{\sin b\ .\ \sin c}}.$$

Or, $\log \sin \frac{1}{2}A = \log \sin(\frac{1}{2}s - b) + \log \sin(\frac{1}{2}s - c) +$ ar. comp. $\log \sin b +$ ar. comp. $\log \sin c$: where $s = a + b + c$.

$$\begin{array}{rlrl}
\log \sin(\frac{1}{2}s - b) = & \log \sin & 43°58'\tfrac{1}{2} = & 9{\cdot}8415749 \\
\log \sin(\frac{1}{2}s - c) = & \log \sin & 12°41'\tfrac{1}{2} = & 9{\cdot}3418385 \\
\text{A.C.} \log \sin b = & \text{A.C.} \log \sin & 83°13' = & 0{\cdot}0030508 \\
\text{A.C.} \log \sin c = & \text{A.C.} \log \sin & 114°30' = & 0{\cdot}0409771 \\
\hline
 & \text{Sum of the four logs} \ldots\ldots & & 19{\cdot}2274413 \\
\text{Half sum} = \log \sin \frac{1}{2}A = & \log \sin & 24°15'\tfrac{1}{2} = & 9{\cdot}6137206
\end{array}$$

Consequently the angle A is 48°31′.

Then, by the common analogy,

$$\begin{array}{rlll}
\text{As, } \sin a \ldots & \sin & 56°40' \ldots & \log = 9{\cdot}9219401 \\
\text{To, } \sin A \ldots & \sin & 48°31' \ldots & \log = 9{\cdot}8745679 \\
\text{So is, } \sin b \ldots & \sin & 83°13' \ldots & \log = 9{\cdot}9969492 \\
\text{To, } \sin B \ldots & \sin & 62°56' \ldots & \log = 9{\cdot}9495770 \\
\text{And so is, } \sin c \ldots & \sin & 114°30' \ldots & \log = 9{\cdot}9590229 \\
\text{To, } \sin C \ldots & \sin & 125°19 \ldots & \log = 9{\cdot}9116507.
\end{array}$$

So that the remaining angles are, $B = 62°56'$, and $C = 125°19'$.

2dly. By way of comparison of methods, let us find the angle A, by the analogies of Napier, according to case 5 table 3. In order to which, suppose a perpendicular demitted from the angle C on the opposite side c. Then shall we have $\tan \frac{1}{2}$ diff. seg. of $c = \dfrac{\tan \frac{1}{2}(b + a)\ .\ \tan \frac{1}{2}(b - a)}{\tan \frac{1}{2}c}$.

This in logarithms, is

$$\begin{array}{rr}
\log \tan \frac{1}{2}(b + a) = \log \tan 69°56'\tfrac{1}{2} = & 10{\cdot}4375601 \\
\log \tan \frac{1}{2}(b - a) = \log \tan 13°16'\tfrac{1}{2} = & 9{\cdot}3727819 \\
\hline
\text{Their sum} = & 19{\cdot}810\ 420 \\
\text{Subtract } \log \tan \frac{1}{2}c = \log \tan 57°15' = & 10\ 1916\ 94 \\
\hline
\text{Rem. log cos dif. seg} = \log \cos 22°34' = & 9{\cdot}6187026
\end{array}$$

Hence, the segments of the base are 79°49′ and 34°41′.

Therefore,

Therefore, since $\cos A = \tan 79°49' \times \cot b$:

To log tan adja. seg. $=$ log tan $79°49' = 10{\cdot}7456257$
Add log tan side b $=$ log tan $83°13' = 9{\cdot}0753563$
The sum, rejecting 10 from the index $= \log\cos A = \log\cos 48°32'$ } $= 9{\cdot}8209820$

The other two angles may be found as before. The preference is, in this case, manifestly due to the former method.

Ex. 9. In an oblique-angled spherical triangle, are given two sides, equal to 114°30′ and 56°40′ respectively, and the angle opposite the former equal to 125°20′; to find the other parts. Ans. Angles 48°30′ and 62°55′; side, 83°12′.

Ex. 10. Given, in a spherical triangle, two angles, equal to 48°30′ and 125°20′, and the side opposite the latter; to find the other parts.

Ans. Side opposite first angle, 56°40′; other side, 83°12′; third angle, 62°54′.

Ex. 11. Given two sides, equal 114°30′ and 56°40′; and their included angle 62°54′: to find the rest.

Ex. 12. Given two angles, 125°20′ and 48°30′, and the side comprehended between them 83°12′: to find the other parts.

Ex. 13. In a spherical triangle, the angles are 48°31′, 62°56′, and 125°20′; required the sides?

Ex. 14. Given two angles, 50°12′, and 58°8′; and a side opposite the former, 62°42′; to find the other parts.

Ans. The third angle is	either 130°56′	or 156°14′.
Side betw. giv. angles,	either 119°4′	or 152°14′.
Side opp. 58°8′,	either 79°12′	or 100°48′.

Ex. 15. The excess of the three angles of a triangle, measured on the earth's surface, above two right angles, is 1 second; what is its area, taking the earth's diameter at 7957^3 miles?

Ans. 76·75299, or nearly $76\frac{3}{4}$ square miles.

Ex. 16. Determine the solid angles of a regular pyramid with hexagonal base, the altitude of the pyramid being to each side of the base, as 2 to 1.

Ans. Plane angle between each two lateral faces 126°52′11″$\frac{1}{2}$.
between the base and each face 66°35′11″$\frac{1}{2}$.
Solid angle at the vertex 89·17111 } The max. angle
Each ditto at the base 218·1905 } being 1000.

CHAPTER

CHAPTER V.

ON GEODESIC OPERATIONS, AND THE FIGURE OF THE EARTH.

SECTION I.

General Account of this kind of Surveying.

ART. 1. In the treatise on Land Surveying in the second volume of this Course of Mathematics, the directions were restricted to the necessary operations for surveying fields, farms, lordships, or at most counties; these being the only operations in which the generality of persons, who practise this kind of measurement, are likely to be engaged: but there are especial occasions when it is requisite to apply the principles of plane and spherical geometry, and the practices of surveying, to much more extensive portions of the earth's surface; and when of course much care and judgment are called into exercise, both with regard to the direction of the practical operations, and the management of the computations. The extensive processes which we are now about to consider, and which are characterised by the terms *Geodesic Operations* and *Trigonometrical Surveying*, are usually undertaken for the accomplishment of one of these three objects. 1. The finding the difference of longitude, between two moderately distant and noted meridians; as the meridians of the observatories at Greenwich and Oxford, or of those at Greenwich and Paris. 2. The accurate determination of the geographical positions of the principal places, whether on the coast or inland, in an island or kingdom; with a view to give greater accuracy to maps, and to accommodate the navigator with the actual position, as to latitude and longitude, of the principal promontories, havens, and ports. These have, till lately, been desiderata, even in this country: the position of some important points, as the Lizard, not being known within seven minutes of a degree; and, until the publication of the Board of Ordnance maps, the best county maps being so erroneous, as in some cases to exhibit *blunders of three miles in distances of less than twenty.*

3. The

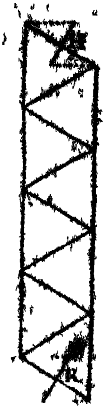
1. The measurement of a degree in various situations, and thence the determination of the figure and magnitude of the earth.

When objects so important as these are to be attained, it is manifest that, in order to ensure the desirable degree of correctness in the results, the instruments employed, the operations performed, and the computations required, must each have the greatest possible degree of accuracy. Of these, the first depend on the artist; the second on the surveyor, or engineer, who conducts them; and the latter on the theorist and calculator: they are these last which will chiefly engage our attention in the present chapter.

2. In the determination of distances of many miles, whether for the survey of a kingdom, or for the measurement of a degree, the whole line intervening between two extreme points is not *absolutely measured;* for this, on account of the inequalities of the earth's surface, would be always very difficult, and often impossible. But, a line of a few miles in length is very carefully measured on some plain, heath, or marsh, which is so nearly level as to facilitate the measurement of an actually horizontal line; and this line being assumed as the base of the operations, a variety of hills and elevated spots are selected, at which signals can be placed, suitably distant and visible one from another: the straight lines joining these points constitute a double series of triangles, of which the assumed base forms the first side; the angles of these, that is, the angles made at each station or signal staff, by two other signal staffs, are carefully measured by a theodolite, which is carried successively from one station to another. In such a series of triangles, care being always taken that one side is common to two of them, all the angles are known from the observations at the several stations; and a side of one of them being given, namely, that of the base measured, the sides of all the rest, as well as the distance from the first angle of the first triangle, to any part of the last triangle, may be found by the rules of trigonometry. And so again, the bearing of any one of the sides, with respect to the meridian, being determined by observation, the bearings of any of the rest, with respect to the same meridian, will be known by computation. In these operations, it is always advisable, when circumstances will admit of it, to measure another base (called a base of verification) at or near the ulterior extremity of the series: for the length of this base, *computed* as one of the sides of the chain of triangles, compared with its length determined by *actual admeasurement*, will be a test of the accuracy of all the operations made in the series between the two bases.

3. Now

3. Now, in every series of triangles, where each angle is to be ascertained with the same instrument, they should, as nearly as circumstances will permit, be equilateral. For, if it were possible to choose the stations in such manner, that each angle should be exactly 60 degrees; then, the half number of triangles in the series, multiplied into the length of one side of either triangle, would, as in the annexed figure, give at once the total distance; and then also, not only the sides of the scale or ladder, constituted by this series of triangles, would be perfectly parallel, but the diagonal steps, marking the progress from one extremity to the other, would be alternately parallel throughout the whole length. Here too, the first side might be found by a base crossing it perpendicularly of about half its length, as at H; and the last side verified by another such base, K, at the opposite extremity. If the respective sides of the series of triangles were 12 or 18 miles, these bases might advantageously be between 6 and 7, or between 9 and 10 miles respectively; according to circumstances. It may also be remarked, (and the reason of it will be seen in the next section) that whenever only two angles of a triangle can be actually observed, each of them should be as nearly as possible 45°, or the sum of them about 90°; for the less the third or computed angle differs from 90°, the less probability there will be of any considerable error. See prob. 1 sect. 2, of this chapter.



4. The student may obtain a general notion of the method, employed in measuring an arc of the meridian, from the following brief sketch and introductory illustrations.

The earth, it is well known, is nearly spherical. It may be either an ellipsoid of revolution, that is, a body formed by the rotation of an ellipse, the ratio of whose axes is nearly that of equality, on one of those axes; or it may approach nearly to the form of such an ellipsoid or spheroid, while its deviations from that form, though small *relatively*, may still be sufficiently great in themselves, to prevent its being called a spheroid with much more propriety than it is called a sphere. One of the methods made use of to determine this point, is by means of extensive Geodesic operations.

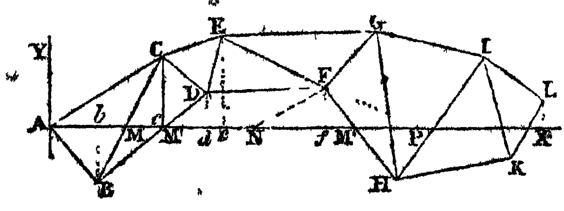
The earth, however, be its exact form what it may, is a planet, which not only revolves in an orbit, but turns upon an axis. Now, if we conceive a plane to pass through the axis of rotation of the earth, and through the zenith of any place on its surface, this plane, if prolonged to the limits of

the apparent celestial sphere, would there trace the circumference of a great circle, which would be the *meridian* of that place. All the points of the earth's surface, which have their zenith in that circumference, will be under the same celestial meridian, and will form the corresponding *terrestrial meridian*. If the earth be an irregular spheroid, this meridian will be a curve of double curvature; but if the earth be a solid of revolution, the terrestrial meridian will be a plane curve.

5. If the earth were a sphere, then every point upon a terrestrial meridian would be at an equal distance from the centre, and of consequence every degree upon that meridian would be of equal length. But if the earth be an ellipsoid of revolution slightly flattened at its poles, and protuberant at the equator; then, as will be shown soon, the degrees of the terrestrial meridian, in receding from the equator towards the poles, will be increased in the duplicate ratio of the right sine of the latitude; and the ratio of the earth's axes, as well as their actual magnitude, may be ascertained by comparing the lengths of a degree on the meridian in different latitudes. Hence appears the great importance of measuring a degree.

6. Now, instead of actually tracing a meridian on the surface of the earth,—a measure which is prevented by the interposition of mountains, woods, rivers, and seas,—a construction is employed which furnishes the same result. It consists in this.

Let ABCDEF &c, be a series of triangles, carried on, as nearly as may be, in the direction of the meridian, according



to the observations in art. 3. These triangles are really *spherical* or spheroidal triangles; but as their curvature is extremely small, they are treated the same as rectilinear triangles, either by reducing them to the *chords* of the respective terrestrial arcs AC, AB, BC, &c, or by deducting *a third* of the excess, of the sum of the three angles of each triangle above two right angles, from each angle of that triangle, and working with the remainders, and the three sides, as the dimensions of a plane triangle; the proper reductions to the centre of the station, to the horizon, and to the level of the sea, having been previously made. These computations being made throughout

throughout the series, the sides of the successive triangles are contemplated as arcs of the terrestrial spheroid. Suppose that we know, by observation, and the computations which will be explained in this chapter, the *azimuth*, or the inclination of the side AC to the first portion AM of the measured meridian, and that we find, by trigonometry, the point M where that curve will cut the side BC. The points A, B, C, being in the same horizontal plane, the line AM will also be in that plane: but, because of the curvature of the earth, the prolongation MM′, of that line, will be found *above* the plane of the second horizontal triangle BCD: if, therefore, without changing the angle CMM′, the line MM′ be brought down to coincide with the plane of this second triangle, by being turned about BC as an axis, the point M′ will describe an arc of a circle, which will be so very small, that it may be regarded as a right line perpendicular to the plane BCD: whence it follows, that the operation is reduced to bending down the side MM′ in the plane of the meridian, and calculating the distance AMM′, to find the position of the point M′. By bending down thus in imagination, one after another, the parts of the meridian on the corresponding horizontal triangles, we may obtain, by the aid of the computation, the direction and the length of such meridian, from one extremity of the series of triangles, to the other.

A line traced in the manner we have now been describing, or deduced from trigonometrical measures, by the means we have indicated, is called a *geodetic* or *geodesic line*: it has the property of being the shortest which can be drawn between its two extremities on the surface of the earth; and it is therefore the proper itinerary measure of the distance between those two points. Speaking rigorously, this curve differs a *little* from the terrestrial meridian, when the earth is not a solid of revolution: yet, in the real state of things, the difference between the two curves is so extremely minute, that it may safely be disregarded.

7. If now we conceive a circle perpendicular to the celestial meridian, and passing through the vertical of the place of the observer, it will represent the prime vertical of that place. The series of all the points of the earth's surface which have their zenith in the circumference of this circle, will form the *perpendicular* to the meridian, which may be traced in like manner as the meridian itself.

In the sphere the perpendiculars to the meridian are great circles which all intersect mutually, on the equator, in two points diametrically opposite: but in the ellipsoid of revolu-

 tion,

tion, and *a fortiori* in the irregular spheroid, these occurring perpendiculars are curves of double curvature. Whatever be the nature of the terrestrial spheroid, the parallels to the equator are curves of which all the points are at the same latitude: on an ellipsoid of revolution, these curves are plane and circular.

8. The situation of a place is determined, when we know either the individual perpendicular to the meridian, or the individual parallel to the equator, on which it is found, and its position on such perpendicular, or on such parallel. Therefore, when all the triangles, which constitute such a series as we have spoken of, have been computed, according to the principles just sketched, the respective positions of their angular points, either by means of their longitudes and latitudes, or of their distances from the first meridian, and from the perpendicular to it. The following is the method of computing these distances.

Suppose that the triangles ABC, BCD, &c, (see the fig. to art. 6) make part of a chain of triangles, of which the sides are arcs of great circles of a sphere, whose radius is the distance from the level or surface of the sea to the centre of the earth; and that we know by observation the angle CAX, which measures the *azimuth* of the side AC, or its inclination to the meridian AX. Then, having found the excess E, of the three angles of the triangle ACc (Cc being perpendicular to the meridian) above two right angles, by reason of a theorem which will be demonstrated in prob. 8 of this chapter, subtract a third of this excess from each angle of the triangle, and thus by means of the following proportions find Ac, and Cc.

$$\sin(90^\circ - \tfrac{1}{3}E) : \cos(CAc - \tfrac{1}{3}E) :: AC : Ac;$$
$$\sin(90^\circ - \tfrac{1}{3}E) : \sin(CAc - \tfrac{1}{3}E) :: AC : Cc.$$

The azimuth of AB is known immediately, because BAX = CAB − CAX; and if the spherical excess proper to the triangle ABM′ be computed, we shall have

$$AM'B = 180^\circ - M'AB - ABM' + E.$$

To determine the sides AM′, BM′, a third of E must be deducted from each of the angles of the triangle ABM′; and then these proportions will obtain: viz,

$$\sin(180^\circ - M'AB - ABM' + \tfrac{2}{3}E) : \sin(ABM' - \tfrac{1}{3}E) :: AB : AM',$$
$$\sin(180^\circ - M'AB - ABM' + \tfrac{2}{3}E) : \sin(M'AB - \tfrac{1}{3}E) :: AB : BM'.$$

In each of the right-angled triangles Abb, Add, are known two angles and the hypothenuse, which is all that is necessary to determine the sides Ab, bB, and Ad, dD. Therefore the distances of the points B, D, from the meridian and from the perpendicular, are known.

9. PRO-

9. Proceeding in the same manner with the triangle ANK, or M [illegible], to obtain [illegible] and [illegible], the prolongation of ED; and therewith the triangle [illegible] to find the side KF and the angles [illegible], [illegible], it will be easy to calculate the rectangular co-ordinates of the point F.

The distance *f*F and the angles DKN, KF*f*, being thus known, we shall have (th. 6 cor. 3 Geom.)

$$fFK = 180° - EKD - DKN - KFf.$$

So that, in the right-angled triangle *f*FK, two angles and one side are known; and therefore the appropriate spherical excess may be computed, and thence the angle *f*K*f* and the sides *f*F, KF. Resolving next the right-angled triangle FKE, we shall in like manner obtain the position of the point E, with respect to the meridian AX, and to its perpendicular AY; that is to say, the distances Ee, and Ae = AF − eF. And thus may the computist proceed through the whole of the series. It is requisite however, previous to these calculations, to draw, by any suitable scale, the chain of triangles observed, in order to see whether any of the subsidiary triangles ACN, NPF, &c, formed to facilitate the computation of the distances from the meridian, and from the perpendicular to it, are too obtuse or too acute.

Such, in few words, is the method to be followed, when we have principally in view the finding the length of the portion of the meridian comprised between any two points, as A and X. It is obvious that, in the course of the computations, the azimuths of a great number of the sides of triangles in the series is determined; it will be easy therefore to check and verify the work in its process, by comparing the azimuths found by observation, with those resulting from the calculations. The amplitude of the whole arc of the meridian measured, is found by ascertaining the *latitude* at each of its extremities; that is, commonly by finding the differences of the zenith distances of some known fixed star, at both those extremities.

10. Some mathematicians, employed in this kind of operations, have adopted different means from the above. They draw through the summits of all the triangles, parallels to the meridian and to its perpendicular; by these means, the sides of the triangles become the hypothenuses of right-angled triangles, which they compute in order, proceeding from some known azimuth, and without regarding the spherical excess, considering all the triangles of the chain as described on a plane surface. This method, however, is manifestly defective in point of accuracy.

Others have computed the sides and angles of all the triangles, by the rules of spherical trigonometry. Others, again, reduce

reduce the observed angles to angles of the chords of the respective arches; and calculate by plane trigonometry, from such reduced angles and their chords. Either of these two methods is equally correct as that by means of the spherical excess; so that the principal reason for preferring one of these to the other, must be derived from its relative facility. As to the methods in which the several triangles are contemplated as spheroidal, they are abstruse and difficult, and may, happily, be safely disregarded; for M. Legendre has demonstrated, in *Mémoires de la Classe des Sciences Physiques et Mathématiques de l'Institut*, 1806, pa. 130, that the difference between spherical and spheroidal angles, is less than *one sixtieth* of a second, in the greatest of the triangles which occurred in the late measurement of an arc of a meridian, between the parallels of Dunkirk and Barcelona.

11. Trigonometrical surveys for the purpose of measuring a degree of a meridian in different latitudes, and thence inferring the figure of the earth, have been undertaken by different philosophers, under the patronage of different governments. As by M. Maupertius, Clairaut, &c, in Lapland, 1736; by M. Bouguer and Condamine, at the equator, 1736—1743; by Cassini, in lat. 45°, 1739—40; by Boscovich and Lemaire, lat. 43°, 1752; by Beccaria, lat. 44°44′, 1768; by Mason and Dixon in America, 1764—8; by Major Lambton, in the East Indies, 1803; by Mechain, Delambre, &c, France, &c, 1790—1805; by Swanberg, Ofverbom, &c, in Lapland, 1802; and by General Roy, Colonel Williams, Mr. Dalby, and Colonel Mudge, in England, from 1784 to the present time. The three last mentioned of these surveys are doubtless the most accurate and important.

The trigonometrical survey in England was first commenced, in conjunction with similar operations in France, in order to determine the difference of longitude between the meridians of the Greenwich and Paris observatories: for this purpose, three of the French Academicians, MM. Cassini, Mechain, and Legendre, met General Roy and Dr. (now Sir Charles) Blagden, at Dover, to adjust their plans of operation. In the course of the survey, however, the English philosophers, selected from the Royal Artillery officers, expanded their views, and pursued their operations, under the patronage, and at the expence of the Honourable Board of Ordnance, in order to perfect the geography of England, and to determine the lengths of as many degrees on the meridian as fell within the compass of their labours.

12. It is not our province to enter into the history of these surveys: but it may be interesting and instructing to speak a

little of the instruments employed, and of the extreme accuracy of some of the results obtained by them.

These instruments are, besides the signals, those for measuring distances, and those for measuring angles. The French philosophers used for the former purpose, in their measurement to determine the length of the *mètre*, rulers of platina and of copper, forming metallic thermometers. The Swedish mathematicians, Swanberg and Ofverbom, employed iron bars, covered towards each extremity with plates of silver. General Roy commenced his measurement of the base at Hounslow-Heath with *deal* rods, each of 20 feet in length. Though they, however, were made of the best seasoned timber, were perfectly straight, and were secured from bending in the most effectual manner; yet the changes in their lengths, occasioned by the variable moisture and dryness of the air, were so great, as to take away all confidence in the results deduced from them. Afterwards, in consequence of having found by experiments, that a solid bar of glass is more dilatable than a tube of the same matter, glass tubes were substituted for the deal rods. They were each 20 feet long, inclosed in wooden frames, so as to allow only of expansion or contraction in length, from heat or cold, according to a law ascertained by experiments. The base measured with these was found to be 27404·08 feet, or about 5·19 miles. Several years afterwards the same base was remeasured by Colonel Mudge, with a steel-chain of 100 feet long, constructed by Ramsden, and jointed somewhat like a watch-chain. This chain was always stretched to the same tension, supported on troughs laid horizontally; and allowances were made for changes in its length by reason of variations of temperature, at the rate of ·0075 of an inch for each degree of heat from 62° of Fahrenheit: the result of the measurement by this chain was found not to differ more than 2¾ inches, from General Roy's determination by means of the glass tubes: a minute difference in a distance of more than 5 miles; which, considering that the measurements were effected by different persons, and with different instruments, is a remarkable confirmation of the accuracy of both operations. And further, as steel chains can be used with more facility and convenience than glass rods, this remeasurement determines the question of the comparative fitness of these two kinds of instruments.

13. For the determination of angles, the French and Swedish philosophers employed *repeating circles* of Borda's construction: instruments which are extremely portable, and with which, though they are not above 14 inches in diameter, the observers can take angles to within 1″ or 2″ of the truth. But

But this kind of instrument, however great its ingenuity in theory, has the accuracy of its observations necessarily limited by the imperfections of the *small* telescope which must be attached to it. General Roy and Colonel Mudge made use of a very excellent theodolite constructed by Ramsden, which, having both an altitude and an azimuth circle, combines the powers of a theodolite, a quadrant, and a transit instrument, and is capable of measuring horizontal angles to fractions of a second. This instrument, besides, has a telescope of a much higher magnifying power than had ever before been applied to observations purely terrestrial; and this is one of the superiorities in its construction, to which is to be ascribed the extreme accuracy in the results of this trigonometrical survey.

Another circumstance which has augmented the accuracy of the English measures, arises from the mode of fixing and using this theodolite. In the method pursued by the Continental mathematicians, a reduction is necessary to the plane of the horizon, and another to bring the observed angles to the true angles at the centres of the signals: these reductions, of course, require formulæ of computation, the actual employment of which *may* lead to error. But, in the trigonometrical survey of England, great care has always been taken to place the centre of the theodolite exactly in the vertical line, previously or subsequently occupied by the centre of the signal: the theodolite is also placed in a perfectly horizontal position. Indeed, as has been observed by a competent judge, "In no other survey has the work in the field been conducted so much with a view to save that in the closet, and at the same time to avoid all those causes of error, however minute, that are not essentially involved in the nature of the problem. The French mathematicians trust to the *correction* of those errors; the English endeavour to *cut them off* entirely; and it can hardly be doubted that the latter, though perhaps the slower and more expensive, is by far the safest proceeding."

14. In proof of the great correctness of the English survey, we shall state a very few particulars, besides what is already mentioned in art. 12.

General Roy, who first measured the base on Hounslow-Heath, measured another on the flat ground of Romney-Marsh in Kent, near the southern extremity of the first series of triangles, and at the distance of more than 60 miles from the first base. The length of this base of verification, as actually measured, compared with that resulting from the computation through the whole series of triangles, differed only by 28 inches.

Colonel Mudge measured another base of verification on Salisbury-

Salisbury-Plain, whose length was 36574·4 feet, or more than 7 miles, the measurement did not differ more than *one inch* from the computation carried through the series of triangles from Hounslow-Heath to Salisbury-Plain. A most remarkable proof of the accuracy with which all the angles, as well as the two bases, were measured!

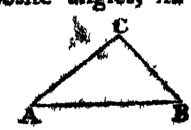
The distance between Beachy-Head in Sussex, and Dunnose in the Isle of Wight, as deduced from a mean of four series of triangles, is 339397 feet, or more than 64¼ miles. The extremes of the four determinations do not differ more than 7 feet, which is less than 1⅓ inches in a mile. Instances of this kind frequently occur in the English survey*. But we have not room to specify more. We must now proceed to discuss the most important problems connected with this subject; and refer those who are desirous to consider it more minutely, to Colonel Mudge's "Account of the Trigonometrical Survey;" Mechain and Delambre, "Base du Systéme Métrique Décimal;" Swanberg, "Exposition des Opérations faites en Lapponie;" and Puissant's works entitled "Geodesie" and "Traité de Topographie, d'Arpentage, &c."

SECTION II.

Problems connected with the detail of Operations in Extensive Trigonometrical Surveys.

PROBLEM I.

It is required to determine the Most Advantageous Conditions of Triangles.



1. In any rectilinear triangle ABC, it is, from the proportionality of sides to the sines of their opposite angles, AB : BC :: sin C : sin A, and consequently AB . sin A = BC . sin C. Let AB be the base, which is supposed to be measured without perceptible error, and which therefore is assumed as constant; then finding the extremely

* Puissant, in his "Geodésie," after quoting some of them, says, "Neanmoins, jusqu'à présent, rien n'égale en exactitude les opérations géodesiques qui ont servi de fondement à notre système métrique." He, however, gives no instances. We have no wish to depreciate the labours of the French measurers; but we cannot yield them the preference on mere assertion.

small

small variation or fluxion of the equation on this hypothesis, it is $AB \cdot \cos A \cdot \dot{A} = \sin C \cdot \dot{BC} + BC \cdot \cos C \cdot \dot{C}$. Here, since we are ignorant of the magnitude of the errors or variations expressed by $\dot{A}$ and $\dot{C}$, suppose them to be equal (a probable supposition, as they are both taken by the same instrument), and each denoted by v: then will

$$\dot{BC} = v \times \frac{AB \cos A - BC \cos C}{\sin C};$$

or, substituting $\frac{BC}{\sin A}$ for its equal $\frac{AB}{\sin C}$, the equation will become $\dot{BC} = v \times (BC \cdot \frac{\cos A}{\sin A} - BC \cdot \frac{\cos C}{\sin C})$;

or, finally, $\dot{BC} = v \cdot BC (\cot A - \cot C)$.

This equation (in the use of which it must be recollected that v taken in seconds should be divided by R″, that is, by the length of the radius expressed in seconds) gives the error $\dot{BC}$ in the estimation of BC occasioned by the errors in the angles A and C. Hence, that these errors, supposing them to be equal, may have no influence on the determination of BC, we must have A = C, for in that case the second member of the equation will vanish.

2. But, as the two errors, denoted by $\dot{A}$, and $\dot{C}$, which we have supposed to be of the same kind, or in the same direction, may be committed in different directions, when the equation will be $\dot{BC} = \pm v \cdot BC (\cot A + \cot C)$; we must enquire what magnitude the angles A and C ought to have, so that the sum of their cotangents shall have the least value possible; for in this state it is manifest that $\dot{BC}$ will have its least value. But, by the formulæ in chap. 3, we have

$$\cot A + \cot C = \frac{\sin (A + C)}{\sin A \cdot \sin C} = \frac{\sin (A + C)}{\frac{1}{2}\cos (A \backsim C) - \frac{1}{2}\cos (A + C)} = \frac{2 \sin B}{\cos (A \backsim C) + \cos B}.$$

Consequently, $\dot{BC} = \pm v \cdot BC \cdot \frac{2 \sin B}{\cos (A \backsim C) + \cos B}$.

And hence, whatever be the magnitude of the angle B, the error in the value of BC will be the least when $\cos (A \backsim C)$ is the greatest possible, which is, when A = C.

We may therefore infer, for a general rule, that *the most advantageous state of a triangle, when we would determine one side only, is when the base is equal to the side sought.*

3. Since, by this rule, the base should be equal to the side sought, it is evident that *when we would determine two sides, the most advantageous condition of a triangle is that it be equilateral.*

4. It

4. It rarely happens, however, that a base can be commodiously measured which is as long as the sides sought. Supposing, therefore, that the length of the base is limited, but that its direction at least may be chosen at pleasure, we proceed to enquire what that direction should be, in the case where one only of the other two sides of the triangles is to be determined.

Let it be imagined, as before, that AB is the base of the triangle ABC, and BC the side required. It is proposed to find the least value of $\cot A \mp \cot C$, when we cannot have $A = C$.

Now, in the case where the negative sign obtains, we have $\cot A - \cot C = \frac{AB - BC \cdot \cos B}{BC \cdot \sin B} - \frac{BC - AB \cdot \cos B}{AB \cdot \sin B} = \frac{AB^2 - BC^2}{AB \cdot BC \cdot \sin B}$. This equation again manifestly indicates the equality of AB and BC, in circumstances where it is possible: but if AB and BC are constant, it is evident, from the form of the denominator of the last fraction, that the fraction itself will be the least, or $\cot A - \cot C$ the least, when $\sin B$ is a maximum, that is, when $B = 90°$.

5. When the positive sign obtains, we have $\cot A + \cot C = \cot A + \frac{\sqrt{(BC^2 - AB^2 \sin^2 A)}}{AB \sin A} = \cot A + \sqrt{(\frac{BC^2}{AB^2 \sin^2 A} - 1)}$. Here, the least value of the expression under the radical sign, is obviously when $A = 90°$. And in that case the first term, $\cot A$, would disappear. Therefore the least value of $\cot A + \cot C$, obtains when $A = 90°$; conformably to the rule given by M. Bouguer (*Fig. de la Terre*, pa. 88). But we have already seen that in the case of $\cot A - \cot C$, we must have $B = 90$. Whence we conclude, since the conditions $A = 90°$, $B = 90°$, cannot obtain simultaneously, that a medium result would give $A = B$.

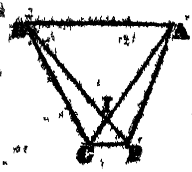
If we apply to the side AC the same reasoning as to BC, similar results will be obtained: therefore, in general, *when the base cannot be equal to one or to both the sides required, the most advantageous condition of the triangle is, that the base be the longest possible, and that the two angles at the base be equal.* These equal angles, however, should never, if possible, be less than 23 degrees.

PROBLEM II.

To deduce, from Angles measured Out of one of the stations, but Near it, the True Angles at the station.

When the centre of the instrument cannot be placed in the vertical line occupied by the axis of a signal, the angles observed must undergo a reduction, according to circumstances.

1. Let



1. Let C be the centre of the station, P the place of the centre of the instrument, or the summit of the observed angle APB: it is required to find C, the measure of ACB, supposing there to be known APB $=$ P, BPC $= p$, CP $= d$, BC $=$ L, AC $=$ R.

Since the exterior angle of a triangle is equal to the sum of the two interior opposite angles (th. 16 Geom.), we have, with respect to the triangle IAP, AIB $=$ P $+$ IAP; and with regard to the triangle BIC, AIB $=$ C $+$ CBP. Making these two values of AIB equal, and transposing IAP, there results

$$\text{C} = \text{P} + \text{IAP} - \text{CBP}.$$

But the triangles CAP, CBP, give

$$\sin \text{CAP} = \sin \text{IAP} = \frac{\text{CP}}{\text{AC}} \sin \text{APC} = \frac{d \cdot \sin (\text{P} + p)}{\text{R}};$$

$$\sin \text{CBP} = \frac{\text{CP}}{\text{BC}} \cdot \sin \text{BPC} = \frac{d \cdot \sin p}{\text{L}}.$$

And, as the angles CAP, CBP, are, by the hypothesis of the problem, always very small, their sines may be substituted for their arcs or measures: therefore

$$\text{C} - \text{P} = \frac{d \cdot \sin (\text{P} + p)}{\text{R}} - \frac{d \cdot \sin p}{\text{L}}.$$

Or, to have the reduction in seconds,

$$\text{C} - \text{P} = \frac{d}{\sin 1''} \left(\frac{\sin (\text{P} + p)}{\text{R}} - \frac{\sin p}{\text{L}}\right).$$

The use of this formula cannot in any case be embarrassing, provided the signs of $\sin p$, and $\sin (\text{P} + p)$ be attended to. Thus, the first term of the correction will be positive, if the angle $(\text{P} + p)$ is comprised between 0 and 180°; and it will become negative, if that angle surpass 180°. The contrary will obtain in the same circumstances with regard to the second term, which answers to the angle of direction p. The letter R denotes the distance of the object A to the right, L the distance of the object B situated to the left, and p the angle at the place of observation, between the centre of the station and the object to the left.

2. An approximate reduction to the centre may indeed be obtained by a single term; but it is not quite so correct as the form above. For, by reducing the two fractions in the second member of the last equation but one to a common denominator, the correction becomes

$$\text{C} - \text{P} = \frac{d\text{L} \cdot \sin (\text{P} + p) - d\text{R} \cdot \sin p}{\text{LR}}.$$

But the triangle ABC gives $\text{L} = \frac{\text{R} \cdot \sin \text{A}}{\sin \text{B}} = \frac{\text{R} \cdot \sin \text{A}}{\sin (\text{A} + \text{C})}$.

And

And because P is always very nearly equal to C, the sine of A + P will differ extremely little from sin (A + C), and may therefore be substituted for it, making $L = \frac{R \sin A}{\sin (A + P)}$.

Hence we manifestly have

$$C - P = \frac{d \cdot \sin A \cdot \sin (P + p) - d \cdot \sin p \cdot \sin (A + P)}{R \cdot \sin A};$$

Which, by taking the expanded expressions for sin (P + p), and sin (A + P), and reducing to seconds, gives

$$C - P = \frac{d}{\sin 1''} \cdot \frac{\sin P \cdot \sin (A - p)}{R \cdot \sin A}.$$

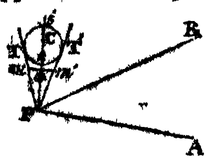
3. When either of the distances R, L, becomes infinite, with respect to d, the corresponding term in the expression art. 1 of this problem, vanishes, and we have accordingly

$$C - P = -\frac{d \cdot \sin p}{L \cdot \sin 1''}, \text{ or } C - P = \frac{d \sin (P + p)}{R \cdot \sin 1''}.$$

The first of these will apply when the object A is a heavenly body, the second when B is one. When both A and B are such, then C − P = 0.

But without supposing either A or B infinite, we may have C − P = 0, or C = P in innumerable instances: that is, in all cases in which the centre P of the instrument is placed in the circumference of the circle that passes through the three points A, B, C; or when the angle BPC is equal to the angle BAC, or to BAC + 180°. Whence, though C should be inaccessible, the angle ACB may commonly be obtained by observation, without any computation. It may further be observed, that when P falls in the circumference of the circle passing through the three points A, B, C, the angles A, B, C, may be determined solely by measuring the angles APB and BPC. For, the opposite angles ABC, APC, of the quadrangle inscribed in a circle, are (theor. 54 Geom.) = 180°. Consequently, ABC = 180° − APC, and BAC = 180° − (ABC + ACB) = 180° − (ABC + APB).

4. If one of the objects, viewed from a further station, be a vane or staff in the centre of a steeple, it will frequently happen that such object, when the observer comes near it, is both invisible and inaccessible. Still there are various methods of finding the exact angle at C. Suppose, for example, the signal-staff be in the centre of a circular tower, and that the angle APB was taken at P near its base. Let the tangents PT, PT′, be marked, and on them two equal and arbitrary distances Pm, Pm', be measured. Bisect mm' at the point n; and, placing there a signal-

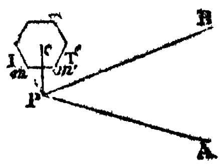


staff,

staff, measure the angle MPB, which, (since PM prolonged obviously passes through C the centre,) will be the angle p of the preceding investigation. Also, the distance Bs added to the radius Os of the tower, will give PC $= d$ in the former investigation.

If the circumference of the tower cannot be measured, and the radius thence inferred, proceed thus: Measure the angles BPT, BPT′, then will BPC $= \frac{1}{2}$(BPT + BPT′) $= p$; and CPT $=$ BPT $-$ BPC: Measure PT, then PC $=$ PT . sec CPT $= d$. With the values of p and d, thus obtained, proceed as before.

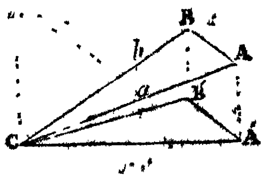
5. If the base of the tower be polygonal and *regular*, as most commonly happens; assume P in the point of intersection of two of the sides prolonged, and BPC′ $= \frac{1}{2}$(BPT + BPT′) as before, PT $=$ the distance from P to the middle of one of the sides whose prolongation passes through P; and hence PC is found, as above. If the figure be a regular hexagon, then the triangle Pmm' is equilateral, and PC $= m'm\sqrt{3}$.



PROBLEM III.

To Reduce angles measured in a Plane Inclined to the horizon, to the Corresponding Angles in the Horizontal Plane.

Let BCA be an angle measured in a plane inclined to the horizon, and let B CA′ be the corresponding angle in the horizontal plane. Let d and d' be the zenith distances, or the complements of the angle of elevation ACA′, BCB′. Then from z the zenith of the observer, or of the angle C, draw the arcs za, zb, of vertical circles, measuring the zenith distances, d, d', and draw the arc ab of another great circle to measure the angle C. It follows from this construction, that the angle z, of the spherical triangle zab, is equal to the horizontal angle A′C′B; and that, to find it, the three sides $za = d$, $zb = d'$, $ab =$ C, are given. Call the sum of these s; then the resulting formulæ of prob. 2 ch. iv, applied to the present instance, becomes



$$\sin\tfrac{1}{2}z = \sin\tfrac{1}{2}C = \sqrt{\frac{\sin\frac{1}{2}(s-d)\,.\,\sin\frac{1}{2}(s-d')}{\sin d\,.\,\sin d'}}.$$

If

If h and h' represent the angles of altitude ACA', BCB', the preceding expression will become

$$\sin \tfrac{1}{2}\text{C} = \sqrt{\frac{\sin \frac{1}{2}(\text{C} + h - h') \cdot \sin \frac{1}{2}(\text{C} + h' - h)}{\cos h \cdot \cos h'}}.$$

Or, in logarithms,

$$\log \sin \tfrac{1}{2}\text{C} = \tfrac{1}{2}(20 + \log \sin \tfrac{1}{2}(\text{C} + h - h') + \log \sin \tfrac{1}{2}(\text{C} + h' - h) - \log \cos h - \log \cos h'.$$

Cor. 1. If $h = h'$, then is $\sin \frac{1}{2}\text{C} = \frac{\sin \frac{1}{2}\text{ACB}}{\cos h}$; and
$\log \sin \frac{1}{2}\text{A'CB'} = 10 + \log \sin \frac{1}{2}\text{ACB} - \log \cos h$.

Cor. 2. If the angles h and h' be very small, and nearly equal; then, since the cosines of small angles vary extremely slowly, we may, without sensible error, take
$\log \sin \frac{1}{2}\text{A'CB'} = 10 + \log \sin \frac{1}{2}\text{ACB} - \log \cos \frac{1}{2}(h + h')$.

Cor. 3. In this case the correction $x = \text{A'CB'} - \text{ACB}$, may be found by the expression

$$x = \sin 1'' (\tan \tfrac{1}{2}\text{C}(\tfrac{1}{4}\text{O} - \frac{d + d'}{2})^2 - \cot \tfrac{1}{2}\text{C}(\frac{d - d'}{2})^2).$$

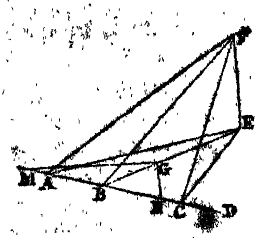
And in this formula, as well as the first given for $\sin \frac{1}{2}\text{C}$, d and d' may be either one or both greater or less than a quadrant; that is, the equations will obtain whether ACA' and BCB' be each an elevation or a depression.

Scholium. By means of this problem, if the altitude of a hill be found barometrically, according to the method described in the 2d volume, or geometrically, according to some of those described in heights and distances, or that given in the following problem; then, finding the angles formed at the place of observation, by any objects in the country below, and their respective angles of depression, their horizontal angles, and thence their distances, may be found, and their relative places fixed in a map of the country; taking care to have a sufficient number of angles between intersecting lines, to verify the operations.

PROBLEM IV.

Given the Angles of Elevation of any Distant object, taken at Three places in a Horizontal Right Line, which does not pass through the point directly below the object; and the Respective Distances between the stations; to find the Height of the Object, and its Distance from either station.

Let AED be the horizontal plane: FE the perpendicular height of the object F above that plane; A, B, C, the three places of observation; FAE, FBE, FCE, the respective angles of



of elevation, and AB, BC, the given distances. Then, since the triangles AEF, BEF, CEF, are all right angled at E; the distances AE, BE, CE, will manifestly be as the cotangents of the angles of elevation at A, B, and C: and we have to determine the point E, so that those lines may have that ratio. To effect this geometrically, use the following

Construction. Take BM, on AC produced, equal to BC, BN equal to AB; and make

MG : BM(= BC) : : cot A : cot B,

and BN(= AB) : NG : : cot B : cot C.

With the lines MN, MG, NG, constitute the triangle MNG; and join BG. Draw AE so, that the angle EAB may be equal to MGB; this line will meet BG produced in E, the point in the horizontal plane falling perpendicularly below F.

Demonstration. By the similar triangles AEB, GMB, we

have AE : BE : : MG : MB : : cot A : cot B,

and BE : BA(= BN) : : BM : BG.

Therefore the triangles BEC, BGN, are similar; consequently BE : EC : : BN : NG : : cot B : cot C. Whence it is obvious that AE, BE, CE, are respectively as cot A, cot B, cot C.

Calculation. In the triangle MGN, all the sides are given, to find the angle GMN = angle AEB. Then, in the triangle MGB, two sides and the included angle are given, to find the angle MGB = angle EAB. Hence, in the triangle AEB, are known AB and all the angles, to find AE, and BE. And then EF = AE . tan A = BE . tan B.

Otherwise, independent of the construction, thus,

Put AB = D, BC = d, EF = x; and then express algebraically the following theorem, given at p. 128 Simpson's Select Exercises:

$$AE^2 . BC + CE^2 . AB = BE^2 . AC + AC . AB . BC,$$

the line EB being drawn from the vertex E of the triangle ACE, to any point B in the base. The equation thence originating is

$$dx^2 . \cot^2 A + Dx^2 . \cot^2 C = (D + d)x^2 . \cot^2 B + (D + d)Dd.$$

And from this, by transposing all the unknown terms to one side, and extracting the root, there results

$$x = \sqrt{\frac{(D+d)Dd}{d . \cot^2 A + D . \cot^2 C - (D + d) \cot^2 B}}.$$

Whence

Whence EF is known, and the distances AE, BE, CE, are readily found.

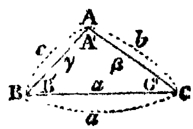
Cor. When $D = d$, or $D + d = 2D = 2d$, the expression becomes better suited for logarithmic computation, being then

$$x = d \div \sqrt{(\tfrac{1}{2}\cot^2 A + \tfrac{1}{2}\cot^2 C - \cot^2 B)}$$

In this case, therefore, the rule is as follows: Double the log. cotangents of the angles of elevation of the extreme stations, find the natural numbers answering thereto, and take half their sum; from which subtract the natural number answering to twice the log. cotangent of the middle angle of elevation: then half the log. of this remainder subtracted from the log. of the measured distance between the 1st and 2d, or the 2d and 3d stations, will be the log. of the height of the object.

PROBLEM V.

In Any Spherical Triangle, knowing Two Sides and the Included Angle; it is required to find the Angle Comprehended by the Chords of those two sides.



Let the angles of the spherical triangle be A, B, C, the corresponding angles included by the chords A′, B′, C′; the spherical sides opposite the former a, b, c, the chords respectively opposite the latter α, β, γ; then, there are given b, c, and A, to find A′.

Here, from prob. 1 equa. I chap. iv, we have

$$\cos a = \sin b \,.\, \sin c \,.\, \cos A + \cos b \,.\, \cos c.$$

But $\cos c = \cos(\tfrac{1}{2}c + \tfrac{1}{2}c) = \cos^2\tfrac{1}{2}c - \sin^2\tfrac{1}{2}c$ (by equa. V ch. iii) $= (1 - \sin^2\tfrac{1}{2}c) - \sin^2\tfrac{1}{2}c = 1 - 2\sin^2\tfrac{1}{2}c$. And in like manner $\cos a = 1 - 2\sin^2\tfrac{1}{2}a$, and $\cos b = 1 - 2\sin^2\tfrac{1}{2}b$. Therefore the preceding equation becomes

$$1 - 2\sin^2\tfrac{1}{2}a = 4\sin\tfrac{1}{2}b \,.\, \cos\tfrac{1}{2}b \,.\, \sin\tfrac{1}{2}c \,.\, \cos\tfrac{1}{2}c \,.\, \cos A + (1 - 2\sin^2\tfrac{1}{2}b) \,.\, (1 - 2\sin^2\tfrac{1}{2}c).$$

But $\sin\tfrac{1}{2}a = \tfrac{1}{2}\alpha$, $\sin\tfrac{1}{2}b = \tfrac{1}{2}\beta$, $\sin\tfrac{1}{2}c = \tfrac{1}{2}\gamma$: which values substituted in the equation, we obtain, after a little reduction,

$$2 \times \frac{\beta^2 + \gamma^2 - \alpha^2}{4} = \beta\gamma \,.\, \cos\tfrac{1}{2}b \,.\, \cos\tfrac{1}{2}c \,.\, \cos A + \tfrac{1}{4}\beta^2\gamma^2.$$

Now, (equa. II ch. iii), $\cos A' = \dfrac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma}$. Therefore, by substitution,

$$\beta\gamma \,.\, \cos A' = \beta\gamma \,.\, \cos\tfrac{1}{2}b \,.\, \cos\tfrac{1}{2}c \,.\, \cos A + \tfrac{1}{4}\beta^2\gamma^2;$$

whence, dividing by $\beta\gamma$, there results

$$\cos A' = \cos\tfrac{1}{2}b \,.\, \cos\tfrac{1}{2}c \,.\, \cos A + \tfrac{1}{2}\beta \,.\, \tfrac{1}{2}\gamma;$$

or, lastly, by restoring the values of $\tfrac{1}{2}\beta$, $\tfrac{1}{2}\gamma$, we have

$$\cos A' = \cos\tfrac{1}{2}b \,.\, \cos\tfrac{1}{2}c \,.\, \cos A + \sin\tfrac{1}{2}b \,.\, \sin\tfrac{1}{2}c \;.\;.\;.\; (\text{I.})$$

 Cor.

Cor. 1. It follows evidently from this formula, that when the spherical angle is right or obtuse, it is always *greater* than the corresponding angle of the chords.

Cor. 2. The spherical angle, if acute, is *less* than the corresponding angle of the chords, when we have cos A greater than $\frac{\sin\frac{1}{2}b \cdot \sin\frac{1}{2}c}{1-\cos\frac{1}{2}b \cdot \cos\frac{1}{2}c}$.

PROBLEM VI.

Knowing Two Sides and the Included Angle of a Rectilinear Triangle, it is required to find the Spherical Angle of the Two Arcs of which those two sides are the chords.

Here β, γ, and the angle A$'$ are given, to find A. Now, since in all cases, $\cos = \sqrt{(1-\sin^2)}$, we have

$$\cos\tfrac{1}{2}b \cdot \cos\tfrac{1}{2}c = \sqrt{[(1-\sin^2\tfrac{1}{2}b) \cdot (1-\sin^2\tfrac{1}{2}c)]};$$

we have also, as above, $\sin\frac{1}{2}b = \frac{1}{2}\beta$, and $\sin\frac{1}{2}c = \frac{1}{2}\gamma$. Substituting these values in the equation I of the preceding problem, there will result, by reduction,

$$\cos A = \frac{\cos A' - \frac{1}{4}\beta\gamma}{\sqrt{(1-\frac{1}{2}\beta) \cdot (1+\frac{1}{2}\beta) \cdot (1-\frac{1}{2}\gamma) \cdot (1+\frac{1}{2}\gamma)}} \ldots \text{(II.)}$$

To compute by this formula, the values of the sides β, γ, must be reduced to the corresponding values of the chords of a circle whose radius is unity. This is easily effected by dividing the values of the sides given in feet, or toises, &c, by such a power of 10, that neither of the sides shall exceed 2, the value of the greatest chord, when radius is equal to unity.

From this investigation, and that of the preceding problem, the following corollaries may be drawn.

Cor. 1. If $c = b$, and of consequence $\gamma = \beta$, then will

$$\cos A' = \cos A \cdot \cos^2\tfrac{1}{2}c + \sin^2\tfrac{1}{2}c; \text{ and thence}$$

$$1-2\sin^2\tfrac{1}{2}A' = (1-2\sin^2\tfrac{1}{2}A)\cos^2\tfrac{1}{2}c + (1-\cos^2\tfrac{1}{2}c):$$

from which may be deduced

$$\sin\tfrac{1}{2}A' = \sin\tfrac{1}{2}A \cdot \cos\tfrac{1}{2}c \ldots \text{(III.)}$$

Cor. 2. Also, since $\cos\frac{1}{2}c = \sqrt{(1-\sin^2\frac{1}{2}c)} = \sqrt{(1-\frac{1}{4}\gamma^2)}$, equa. II will, in this case, reduce to

$$\sin\tfrac{1}{2}A = \frac{\sin\frac{1}{2}A'}{\sqrt{(1-\frac{1}{2}\gamma) \cdot (1+\frac{1}{2}\gamma)}} \ldots \text{(IV.)}$$

Cor. 3. From the equation III, it appears that the vertical angle of an isosceles spherical triangle, is always *greater* than the corresponding angle of the chords.

Cor. 4. If $A = 90°$, the formulæ I, II, give

$$\cos A' = \sin\tfrac{1}{2}b \cdot \sin\tfrac{1}{2}c = \tfrac{1}{4}\beta\gamma. \ldots \text{(V.)}$$

These five formulæ are strict and rigorous, whatever be the magnitude of the triangle. But if the triangles be small, the arcs may be put instead of the sines in equa. V, then

Cor. 5. As $\cos A' = \sin(90° - A') =$ in this case, $90° - A'$; the small excess of the spherical right angle over the corresponding

sponding rectilinear angle, will, supposing the arcs b, c, taken in seconds, be given in seconds by the following expression,

$$90° - A' = \frac{\frac{1}{4}bc}{R''} = \frac{bc}{4R''} \dots (VI.)$$

The error in this formula will not amount to a second, when $b + c$ is less than 10°, or than 700 miles measured on the earth's surface.

Cor. 6. If the hypothenuse does not exceed $1\frac{1}{2}°$, we may substitute a sin C instead of c, and a cos C instead of b; this will give $bc = a^2 . \sin C . \cos C = \frac{1}{2}a^2 . \sin 2(90° - B) = \frac{1}{2}a^2 . \sin 2B$: whence

$$(90° - A') = \frac{a^2 . \sin 2C}{8R''} = \frac{a^2 . \sin 2B}{8R''} \dots (VII.)$$

If $a = 1\frac{1}{2}°$, and $B = C = 45°$ nearly; then will $90° - A' = 17''\cdot7$.

Cor. 7. Retaining the same hypothesis of $A = 90°$, and $a =$ or $< 1\frac{1}{2}°$, we have

$$B - B' = \frac{b^2 . \cot B}{8R''} = \frac{bc}{8R''} \dots (VIII.)$$

$$\text{Also } C - C' = \frac{bc}{8R''} \dots (IX.)$$

Cor. 8. Comparing formulæ VIII, IX, with VI, we have $B - B' = C - C' = \frac{1}{2}(90° - A')$. Whence it appears that the sum of the two excesses of the oblique spherical angles, over the corresponding angles of the chords, in a small right-angled triangle, is equal to the excess of the right angle over the corresponding angle of the chords. So that either of the formulæ VI, VII, VIII, IX, will suffice to determine the difference of each of the three angles of a small right-angled spherical triangle, from the corresponding angles of the chords. And hence *this* method may be applied to the measuring an arc of the meridian by means of a series of triangles. See arts. 8, 9, sect. 1 of this chapter.

PROBLEM VII.

In a Spherical Triangle ABC, Right Angled at A, knowing the Hypothenuse BC (*less than* 4°) and the Angle B, it is required to find the Error e committed through finding by Plane Trigonometry, the Opposite Side AC.

Referring still to the diagram of prob. 5, where we now suppose the spherical angle A to be right, we have (theor. 10 chap. iv) sin b = sin a . sin B. But it has been remarked at pa. 5 vol. ii, that the sine of any arc A is equal to the sum of the following series;

$$\sin A = A - \frac{A^3}{2.3} + \frac{A^5}{2.3.4.5} - \frac{A^7}{2.3.4.5.6.7} + \&c.$$

$$\text{or, } \sin A = A - \frac{A^3}{6} + \frac{A^5}{120} - \frac{A^7}{5040} + \&c.$$

 And,

And, in the present enquiry, all the terms after the second may be neglected, because the 5th power of an arc of 4° divided by 120, gives a quotient not exceeding 0″·01. Consequently, we may assume $\sin b = b - \frac{1}{6}b^3$, $\sin a = a - \frac{1}{6}a^3$; and thus the preceding equation will become,

$$b - \tfrac{1}{6}b^3 = \sin B(a - \tfrac{1}{6}a^3)$$

$$\text{or, } b = a \cdot \sin B - \tfrac{1}{6}(a^3 \cdot \sin B - b^3).$$

Now, if the triangle were considered as rectilinear, we should have $b = a \cdot \sin B$; a theorem which manifestly gives the side b or AC too great by $\frac{1}{6}(a^3 \cdot \sin B - b^3)$. But, neglecting quantities of the fifth order, for the reason already assigned, the last equation but one gives $b^3 = a^3 \cdot \sin^3 B$. Therefore, by substitution, $e = -\frac{1}{6}a^3 \cdot \sin B(1 - \sin^2 B)$: or, to have this error in seconds, take R'' = the radius expressed in seconds, so shall $e = -a \cdot \sin B \cdot \frac{a^2 \cdot \cos^2 B}{6R''R''}$.

Cor. 1. If $a = 4°$, and $B = 35°16'$, in which case the value of $\sin B \cdot \cos^2 B$ is a maximum, we shall find $e = -4\frac{1}{2}''$.

Cor. 2. If, with the same data, the correction be applied, to find the side c adjacent to the given angle, we should have

$$e' = a \cdot \cos B \cdot \frac{a^2 \cdot \sin^2 B}{3R'' R''}.$$

So that this error exists in a contrary sense to the other; the one being subtractive, the other additive.

Cor. 3. The data being the same, if we have to find the angle C, the error to be corrected will be

$$e'' = a^2 \cdot \frac{\sin 2B}{4R''}.$$

As to the excess of the arc over its chord, it is easy to find it correctly from the expressions in prob. 5: but for arcs that are very small, compared with the radius, a near approximation to that excess will be found in the same measures as the radius of the earth, by taking $\frac{1}{24}$ of the quotient of the cube of the length of the arc divided by the square of the radius.

PROBLEM VIII.

It is required to Investigate a Theorem, by means of which, Spherical Triangles, whose Sides are Small compared with the radius, may be solved by the rules for Plane Trigonometry, without considering the Chords of the respective Arcs or Sides.

Let a, b, c, be the sides, and A, B, C, the angles of a spherical triangle, on the surface of a sphere whose radius is r; then

then a similar triangle on the surface of a sphere whose radius $= 1$, will have for its sides $\frac{a}{r}$, $\frac{b}{r}$, $\frac{c}{r}$; which, for the sake of brevity, we represent by α, β, γ, respectively: then, by equa. I chap. iv, we have $\cos A = \frac{\cos\alpha - \cos\beta . \cos\gamma}{\sin\beta . \sin\gamma}$.

Now, r being very great with respect to the sides a, b, c, we may, as in the investigation of the last problem, omit all the terms containing higher than 4th powers, in the series for the sine and cosine of an arc, given at pa. 5 vol. ii: so shall we have, without perceptible error,

$$\cos\alpha = 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{2.3.4} \ldots \sin\beta = \beta - \frac{\beta^3}{2.3}.$$

And similar expressions may be adopted for $\cos\beta$, $\cos\gamma$, $\sin\gamma$. Thus, the preceding equation will become

$$\cos A = \frac{\frac{1}{2}(\beta^2 + \gamma^2 - \alpha^2) + \frac{1}{24}(\alpha^4 - \beta^4 - \gamma^4) - \frac{1}{4}\beta^2\gamma^2}{\beta\gamma(1 - \frac{1}{6}\beta^2 - \frac{1}{6}\gamma^2)}.$$

Multiplying both terms of this fraction by $1 + \frac{1}{6}(\beta^2 + \gamma^2)$, to simplify the denominator, and reducing, there will result,

$$\cos A = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} + \frac{\alpha^4 + \beta^4 + \gamma^4 - 2\alpha^2\beta^2 - 2\alpha^2\gamma^2 - 2\beta^2\gamma^2}{24\beta\gamma}.$$

Here, restoring the values of α, β, γ, the second member of the equation will be entirely constituted of like combinations of the letters, and therefore the whole may be represented by

$$\cos A = \frac{M}{2bc} + \frac{N}{24bcr^2} \ldots (1.)$$

Let, now, A' represent the angle opposite to the side a, in the rectilinear triangle whose sides are equal in length to the arcs a, b, c; and we shall have

$$\cos A' = \frac{b^2 + c^2 - a^2}{2bc} = \frac{M}{2bc}.$$

Squaring this, and substituting for $\cos^2 A'$ its value $1 - \sin^2 A'$, there will result

$$-4b^2c^2 \sin^2 A' = a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 = N.$$

So that, equa. I, reduces to the form

$$\cos A = \cos A' - \frac{bc}{6r^2} \sin^2 A'.$$

Let $A = A' + x$, then, as x is necessarily very small, its second power may be rejected, and we may assume $\cos A = \cos A' - x . \sin A'$: whence, substituting for $\cos A$ this value of it, we shall have $x = \frac{bc}{6r^2} \sin A'$.

It hence appears that x is of the second order, with respect to $\frac{b}{r}$ and $\frac{c}{r}$; and of course that the result is exact to quantities within the fourth order. Therefore, because $A = A' + x$,

$$A = A' + \frac{bc}{6r^2} . \sin A'.$$

But,

But, by prob. 2 rule 2, Mensuration of Planes, $\frac{1}{2}bc$ sin A′ is the area of the rectilinear triangle, whose sides are a, b, and c.

$$\text{Therefore } A = A' + \frac{\text{area}}{3r^2};$$

$$\text{or } A' = A - \frac{\text{area}}{3r^2}.$$

$$\text{In like manner} \left\{ \begin{array}{l} B' = B - \dfrac{\text{area}}{3r^2}. \\ C' = C - \dfrac{\text{area}}{3r^2}. \end{array} \right.$$

$$\text{And } A' + B' + C' = 180° = A + B + C - \frac{\text{area}}{r^2}:$$

$$\text{or, } \frac{\text{area}}{r^2} = A + B + C - 180°.$$

Whence, since the spherical excess is a measure of the area (th. 5 ch. iv), we have this theorem: viz.

A spherical triangle being proposed, of which the sides are very small, compared with the radius of the sphere; if from each of its angles one third of the excess of the sum of its three angles above two right angles be subtracted, the angles so diminished may be taken for the angles of a rectilinear triangle, whose sides are equal in length to those of the proposed spherical triangle.*

Scholium.

We have already given, at th. 5 chap. iv, expressions for finding the spherical excess, in the two cases, where two sides and the included angle of a triangle are known, and where the three sides are known. A few additional rules may with propriety be presented here.

1. The spherical excess E, may be found in seconds, by the expression $E = \frac{R''s}{r}$; where s is the surface of the triangle $= \frac{1}{2}bc \cdot \sin A = \frac{1}{2}ab \cdot \sin C = \frac{1}{2}ac \cdot \sin B = \frac{1}{2}a^2 \cdot \frac{\sin B \cdot \sin C}{\sin(B + C)}$, r is the radius of the earth, in the same measures as a, b, and c, and $R'' = 206264''\cdot8$, the seconds in an arc equal in length to the radius.

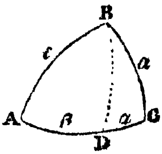
If this formula be applied logarithmically; then log $R'' = \log \frac{1}{\text{arc } 1''} = 5\cdot3144251$.

* [illegible] re[illegible]s [illegible] t [illegible] nounced by M. Legendre, in the [illegible] 87 Legendre's investigation is nearly [illegible] investigation is given by Swanberg, at p. 40, [illegible] sur Lapponie;" but it is defective in [illegible]

2. From

2. From the logarithm of the area of the triangle, taken as a plane one, in feet, subtract the constant log 9·3267737, then the remainder is the logarithm of the excess above 180°, in seconds nearly *.

3. Since $S = \frac{1}{2}bc \,.\, \sin A$, we shall manifestly have $E = \frac{R''}{2r^2}\, bc \,.\, \sin A$. Hence, if from the vertical angle B we demit the perpendicular BD upon the base AC, dividing it into the two segments α, β, we shall have $b = \alpha + \beta$, and thence $E = \frac{R''}{2r^2}\, c(\alpha + \beta) \sin A = \frac{R''}{2r^2}\, \alpha c \,.$ $\sin A + \frac{R''}{2r^2}\, \beta c \,.\, \sin A$. But the two right-angled triangles ABD, CBD, being nearly rectilinear, give $\alpha = a \,.\, \cos C$, and $\beta = c \,.\, \cos A$; whence we have



$$E = \frac{R''}{2r^2}\, ac \,.\, \sin A \,.\, \cos C + \frac{R''}{2r^2}\, c^2 \,.\, \sin A \,.\, \cos A.$$

In like manner, the triangle ABC, which itself is so small as to differ but little from a plane triangle, gives $c \,.\, \sin A = a \,.\, \sin C$. Also, $\sin A \,.\, \cos A = \frac{1}{2} \sin 2A$, and $\sin C \,.\, \cos C = \frac{1}{2} \sin 2C$ (equa. xv. ch. iii). Therefore, finally,

$$E = \frac{R''}{4r^2}\, a^2 \,.\, \sin 2C + \frac{R''}{4r^2}\, c^2 \,.\, \sin 2A.$$

From this theorem a table may be formed, from which the spherical excess may be found; entering the table with each of the sides above the base and its adjacent angle, as arguments.

4. If the base b, and height h, of the triangle are given, then we have evidently $E = \frac{1}{2} bh \frac{R''}{r^2}$. Hence results the following simple logarithmic rule: Add the logarithm of the base of the triangle, taken in feet, to the logarithm of the perpendicular, taken in the same measure; deduct from the sum the logarithm 9·6278037; the remainder will be the common logarithm of the spherical excess in seconds and decimals.

5. Lastly, when the three sides of the triangle are given in feet; add to the logarithm of half their sum, the logs. of the three differences of those sides and that half sum, divide the total of these 4 logs. by 2, and from the quotient subtract the log. 9·3267737; the remainder will be the logarithm of the spherical excess in seconds &c, as before.

One or other of these rules will apply to all cases in which the spherical excess will be required.

* This is General Roy's rule given in the Philosophical Transactions, for 1790, p. 171.

PROBLEM IX.

Given the Measure of a Base on any Elevated Level; to find its Measure when Reduced to the Level of the Sea.

Let r represent the radius of the earth, or the distance from its centre to the surface of the sea, $r + h$ the radius referred to the level of the base measured, the altitude h being determined by the rule for the measurement of such altitudes by the barometer and thermometer, (p. 255 vol. ii, 6th edition); let B be the length of the base measured at the elevation h, and b that of the base referred to the level of the sea. Then because the measured base is all along reduced to the horizontal plane, the two, B and b, will be concentric and similar arcs, to the respective radii $r + h$ and r. Therefore, since similar arcs, whether of spheres or spheroids, are as their radii of curvature, we have

$$r + h : r :: \text{B} : b = \frac{r\text{B}}{r+h}.$$

Hence, also $\text{B} - b = \text{B} - \frac{r\text{B}}{r+h} = \frac{\text{B}h}{r+h}$; or, by actually dividing Bh by $r + h$, we shall have

$$\text{B} - b = \text{B} \times \left(\frac{h}{r} - \frac{h^2}{r^2} + \frac{h^3}{r^3} - \frac{h^4}{r^4} + \&\text{c.}\right)$$

Which is an *accurate* expression for the excess of B above b.

But the mean radius of the earth being more than 21 million feet; if h the difference of level were 50 feet, the second and all succeeding terms of the series could never exceed the fraction $\frac{1}{176000000000}$; and may therefore safely be neglected: so that for all practical purposes we may assume $\text{B} - b = \frac{\text{B}h}{r}$. Or, in logarithms, add the logarithm of the measured base in feet, to the logarithm of its height above the level of the sea, subtract from the sum the logarithm 7·3223947, the remainder will be the logarithm of a number, which taken from the measured base, will leave the reduced base required.

PROBLEM X.

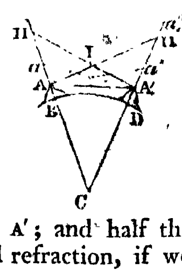
To determine the Horizontal Refraction.

1. Particles of light, in passing from any object through the atmosphere, or part of it, to the eye, do not proceed in a right line; but the atmosphere being composed of an infinitude of strata (if we may so call them) whose density increases as they are posited nearer the earth, the luminous rays which

pass

pass through it are acted on as if they passed successively through media of increasing density, and are therefore inflected more and more towards the earth as the density augments. In consequence of this it is, that rays from objects, whether celestial or terrestrial, proceed in curves which are *concave* towards the earth; and thus it happens, since the eye always refers the place of objects to the direction in which the rays reach the eye, that is, to the direction of the tangent to the curve at that point, that the apparent, or observed elevations of objects, are always *greater* than the true ones. The difference of these elevations, which is, in fact, the *effect* of refraction, is, for the sake of brevity, called *refraction*: and it is distinguished into two kinds, *horizontal* or *terrestrial* refraction, being that which affects the altitudes of hills, towers, and other objects on the earth's surface; and *astronomical* refraction, or that which is observed with regard to the altitudes of heavenly bodies. Refraction is found to vary with the state of the atmosphere, in regard to heat or cold, humidity or dryness, &c: so that, determinations obtained for one state of the atmosphere, will not answer correctly for another, without modification. Tables commonly exhibit the refraction at different altitudes, for some assumed mean state.

2. With regard to the *horizontal* refraction, the following method of determining it has been successfully practised in the English Trigonometrical Survey.



Let A, A′, be two elevated stations on the surface of the earth, BD the intercepted arc of the earth's surface, C the earth's centre, AH′, A′H, the horizontal lines at A, A′, produced to meet the opposite vertical lines CH′, CH. Let a, a', represent the apparent places of the objects A, A′, then is a'AA′ the refraction observed at A, and aA′A the refraction observed at A′; and half the sum of those angles will be the horizontal refraction, if we assume it equal at each station.

Now, an instrument being placed at each of the stations A, A′, the reciprocal observations are made at the same instant of time, which is determined by means of signals or watches previously regulated for that purpose: that is, the observer at A takes the apparent depression of A′, at the same moment that the other observer takes the apparent depression of A.

In the quadrilateral ACA′I, the two angles A, A′ are right angles, and therefore the angles I and C are together equal to two right angles: but the three angles of the triangle IAA′ are

are together equal to two right angles; and consequently the angles A and A′ are together equal to the angle C, which is measured by the arc BD. If therefore the sum of the two depressions HA′a, H′Aa', be taken from the sum of the angles HA′A, H′AA′, or, which is equivalent, from the angle C, (which is known, because its measure BD is known); the remainder is the sum of both refractions, or angles aA′A, a'AA′. Hence this rule, *take the sum of the two depressions from the measure of the intercepted terrestrial arc, half the remainder is the refraction.*

3. If, by reason of the minuteness of the contained arc BD, one of the objects, instead of being depressed, appears elevated, as suppose A′ to a'': then the sum of the angles a''AA′ and aA′A will be greater than the sum IAA′ + IA′A, or than C, by the angle of elevation a''AA′; but if from the former sum there be taken the depression HA′A, there will remain the sum of the two refractions. So that in this case the rule becomes as follows: *take the depression from the sum of the contained arc and elevation, half the remainder is the refraction.*

4. The quantity of this terrestrial refraction is estimated by Dr. Maskelyne at one-tenth of the distance of the object observed, expressed in degrees of a great circle. So, if the distance be 10000 fathoms, its 10th part, 1000 fathoms, is the 60th part of a degree of a great circle on the earth, or 1′, which therefore is the refraction in the altitude of the object at that distance.

But M. Legendre is induced, he says, by several experiments, to allow only $\frac{1}{14}$th part of the distance for the refraction in altitude. So that, on the distance of 10000 fathoms, the 14th part of which is 714 fathoms, he allows only 44″ of terrestrial refraction, so many being contained in the 714 fathoms. See his Memoir concerning the Trigonometrical operations, &c.

Again, M. Delambre, an ingenious French astronomer, makes the quantity of the terrestrial refraction to be the 11th part of the arch of distance. But the English measurers, especially Col. Mudge, from a multitude of exact observations, determine the quantity of the medium refraction to be the 12th part of the said distance.

The quantity of this refraction, however, is found to vary considerably, with the different states of the weather and atmosphere, from the $\frac{1}{7}$th to the $\frac{1}{18}$th of the contained arc. See Trigonometrical Survey, vol. 1 pa. 160, 355.

Scholium.

Scholium.

Having given the mean results of observations on the terrestrial refraction, it may not be amiss, though we cannot enter at large into the investigation, to present here a correct table of mean astronomical refractions. The table which has been most commonly given in books of astronomy, is Dr. Bradley's, computed from the rule $r = 57'' \times \cot (a + 3r)$, where a is the altitude, r the refraction, and $r = 2'35''$ when $a = 20°$. But it has been found by numerous observations, that the refractions thus computed are rather too *small.*—Laplace, in his Mecanique Celeste (tome iv pa. 27) deduces a formula which is strictly similar to Bradley's; for it is $r = m \times \tan (z - nr)$, where z is the zenith distance, and m and n are two constant quantities to be determined from observation. The only advantage of the formula given by the French philosopher, over that given by the English astronomer, is, that Laplace and his colleagues have found more correct coefficients than Bradley had.

Now, if R = $57°{\cdot}2957795$, the arc equal to the radius, if we make $m = \frac{k\text{R}}{n}$, (where k is a constant coefficient which, as well as n, is an abstract number,) the preceding equation will become $\frac{nr}{\text{R}} = k \times \tan (z - nr)$. Here, as the refraction r is always very small, as well as the correction nr, the trigonometrical tangent of the arc nr may be substituted for $\frac{nr}{\text{R}}$; thus we shall have $\tan nr = k \,.\, \tan (z - nr)$.

But $nr = \frac{1}{2}z - (\frac{1}{2}z - nr) \ldots z - nr = \frac{1}{2}z + (\frac{1}{2}z - nr)$;

$$\text{Conseq.}\ \frac{\tan nr}{\tan (z - nr)} = \frac{\tan\left(\frac{z}{2} - \frac{z - 2nr}{2}\right)}{\tan\left(\frac{z}{2} + \frac{z - 2nr}{2}\right)} = \frac{\sin z - \sin (z - 2nr)}{\sin z + \sin (z - 2nr)} = k.$$

Hence, $\sin (z - 2nr) = \frac{1 - k}{1 + k} \,.\, \sin z.$

This formula is easy to use, when the coefficients n and $\frac{1 - k}{1 + k}$ are known: and it has been ascertained, by a mean of many observations, that these are 4 and $\cdot 99765175$ respectively. Thus Laplace's equation becomes

$$\sin (z - 8r) = \cdot 99765175 \sin z:$$

and from this the following table has been computed. Besides the refractions, the differences of refraction, for every 10 minutes of altitude, are given; an addition which will render the table more extensively useful in all cases where great accuracy is required.

Table

Table of Refractions.

Barom. 29·92 inc. Fah. Thermom. 54°.

Alt. app.	Refract.	Diff. on 10′	Alt. app.	Refrac.	Diff. 10′.	Alt. app.	Refrac	Diff. 10′.	Alt. app.	Refr.	Diff 10′.
D. M	M S.	S.	D. M	M S.	S.	D.	M. S.	S.	D.	S.	S.
0 0	33 46·3	112·0	7 0	7 24·8	9·5	14	3 49·8	2·38	56	39·3	0·25
10	31 54·3	105·0	10	7 15·3	9·0	15	3 34·3	2·28	57	37·8	0·24
20	30 9·3	97·3	20	7 6·3	8·6	16	3 20·6	2·02	58	36·4	0·24
30	28 32·1	89·8	30	6 57·7	8·1	17	3 8·5	1·82	59	35·0	0·23
40	27 2·2	83·6	40	6 49·6	7·7	18	2 57·6	1·65	60	33·6	0·22
50	25 38·6	77·4	50	6 41·9	7·5	19	2 47·7	1·48	61	32·3	0·22
1 0	24 21·2	71·6	8 0	6 34·4	7·3	20	2 38·8	1·37	62	31·0	0·21
10	23 9·6	66·2	10	6 27·1	7·1	21	2 30·6	1·24	63	29·7	0·21
20	22 3·4	61·5	20	6 20·0	6·9	22	2 23·2	1·11	64	28·4	0·20
30	21 1·9	57·1	30	6 13·1	6·7	23	2 16·5	1·05	65	27·2	0·20
40	20 4·8	53·3	40	6 6·4	6·5	24	2 10·2	0·98	66	25·9	0·20
50	19 11·5	49·3	50	5 59·9	6·3	25	2 4·3	0·90	67	24·7	0·20
2 0	18 22·2	45·9	9 0	5 53·6	6·2	26	1 58·9	0·83	68	23·5	0·20
10	17 36·3	43·1	10	5 47·4	5·9	27	1 53·9	0·78	69	22·4	0·20
20	16 53·2	39·8	20	5 41·5	5·7	28	1 49·2	0·73	70	21·2	0·20
30	16 13·4	37·4	30	5 35·8	5·5	29	1 44·8	0·70	71	20·0	0·19
40	15 36·0	35·1	40	5 30·3	5·3	30	1 40·6	0·65	72	18·9	0·18
50	15 0·9	32·8	50	5 25·0	5·2	31	1 36·7	0·60	73	17·8	0·18
3 0	14 28·1	30·8	10 0	5 19·8	5·1	32	1 33·1	0·58	74	16·7	0·18
10	13 57·3	28·8	10	5 14·7	5·0	33	1 29·6	0·56	75	15·6	0·18
20	13 28·5	27·2	20	5 9·7	4·8	34	1 26·2	0·55	76	14·5	0·17
30	13 1·3	25·7	30	5 4·9	4·6	35	1 23·1	0·50	77	13·5	0·17
40	12 35·6	24·3	40	5 0·3	4·4	36	1 20·1	0·48	78	12·4	0·17
50	12 11·3	23·0	50	4 55·9	4·2	37	1 17·2	0·47	79	11·3	0·17
4 0	11 48·3	21·7	11 0	4 51·7	4·1	38	1 14·4	0·43	80	10·3	0·17
10	11 26·6	20·5	10	4 47·6	4·0	39	1 11·8	0·42	81	9·2	0·17
20	11 6·1	19·4	20	4 43·6	4·0	40	1 9·3	0·40	82	8·2	0·17
30	10 46·7	18·4	30	4 39·6	3·9	41	1 6·9	0·38	83	7·2	0·17
40	10 28·3	17·4	40	4 35·7	3·9	42	1 4·6	0·37	84	6·1	0·17
50	10 10·9	16·6	50	4 31·8	3·8	43	1 2·4	0·35	85	5·1	0·17
5 0	9 54·3	15·9	12 0	4 28·0	3·7	44	1 0·3	0·34	86	4·1	0·17
10	9 38·4	15·0	10	4 24·3	3·6	45	0 58·2	0·33	87	3·1	0·17
20	9 23·4	14·4	20	4 20·7	3·5	46	0 56·2	0·32	88	2·0	0·17
30	9 9·0	13·7	30	4 17·2	3·4	47	0 54·3	0·31	89	1·0	0·17
40	8 55·3	13·0	40	4 13·8	3·2	48	0 52·4	0·30	90	0·0	
50	8 42·3	12·4	50	4 10·6	3·1	49	0 50·6	0·29			
6 0	8 29·9	11·8	13 0	4 7·5	3·1	50	0 48·9	0·28			
10	8 18·1	11·5	10	4 4·4	3·0	51	0 47·2	0·27			
20	8 6·6	11·0	20	4 1·4	3·0	52	0 45·5	0·26			
30	7 55·6	10·6	30	3 58·4	2·9	53	0 43·9	0·26			
40	7 45·0	10·3	40	3 55·5	2·9	54	0 42·3	0·25			
50	7 34·7	9·9	50	3 52·6	2·8	55	0 40·8	0·25			
7 0	7 24·8		14 0	3 49·8		56	0 39·3				

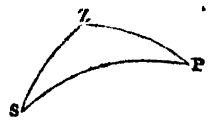
PROBLEM

PROBLEM XI.

To find the Angle made by a Given Line with the Meridian.

1. The easiest method of finding the angular distance of a given line from the meridian, is to measure the greatest and the least angular distance of the vertical plane in which is the star marked α in Ursa minor (commonly called the *pole star*), from the said line: for half the sum of these two measures will manifestly be the angle required.

2. Another method is to observe when the sun is on the given line; to measure the altitude of his centre at that time, and correct it for refraction and parallax. Then, in the spherical triangle ZPS, where Z is the zenith of the place of observation, P the elevated pole, and S the centre of the sun, there are supposed given ZS the zenith distance, or co-altitude of the sun, PS the co-declination of that luminary, PZ the co-latitude of the place of observation, and ZPS the hour angle, measured at the rate of 15° to an hour, to find the angle SZP between the meridian PZ and the vertical ZS, on which the sun is at the given time. And here, as three sides and one angle are known, the required angle is readily found, by saying, as sine ZS : sine ZPS : : sine PS : sine PZS; that is, as the cosine of the sun's altitude, is to the sine of the hour angle from noon; so is the cosine of the sun's declination, to the sine of the angle made by the given vertical and the meridian.



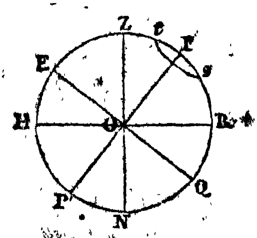
Note. Many other methods are given in books of Astronomy; but the above are sufficient for our present purpose. The first is independent of the latitude of the place; the second requires it.

PROBLEM XII.

To find the Latitude of a Place.

The latitude of a place may be found by observing the greatest and least altitude of a circumpolar star, and then applying to each the correction for refraction; so shall half the sum of the altitudes, thus corrected, be the altitude of the pole, or the latitude.

For,



For, if P be the elevated pole, st the circle described by the star, PR = EZ the latitude: then since $Ps = Pt$, PR must be $= \frac{1}{2}(Rt + Rs)$.

This method is obviously independent of the declination of the star: it is therefore most commonly adopted in trigonometrical surveys, in which the telescopes employed are of such power as to enable the observer to see stars in the day-time: the pole-star being here also made use of.

Numerous other methods of solving this problem likewise are given in books of Astronomy; but they need not be detailed here.

Corol. If the mean altitude of a circumpolar star be thus measured, at the two extremities of any arc of a meridian, the difference of the altitudes will be the measure of that arc: and if it be a small arc, one for example not exceeding a degree of the terrestrial meridian, since such small arcs differ extremely little from arcs of the circle of curvature at their middle points, we may, by a simple proportion, infer the length of a degree whose middle point is the middle of that arc.

Scholium.

Though it is not consistent with the purpose of this chapter to enter largely into the doctrine of astronomical spherical problems; yet it may be here added, for the sake of the young student, that if a = right ascension, d = declination, l = latitude, λ = longitude, p = angle of position (or, the angle at a heavenly body formed by two great circles, one passing through the pole of the equator and the other through the pole of the ecliptic), i = inclination or obliquity of the ecliptic, then the following equations, most of which are new, obtain generally, for all the stars and heavenly bodies.

1. $\tan a = \tan\lambda \,.\, \cos i - \tan l \,.\, \sec\lambda \,.\, \sin i.$
2. $\sin d = \sin\lambda \,.\, \cos l \,.\, \sin i + \sin l \,.\, \cos i.$
3. $\tan\lambda = \sin i \,.\, \tan d \,.\, \sec a + \tan a \,.\, \cos i.$
4. $\sin l = \sin d \,.\, \cos i - \sin a \,.\, \cos d \,.\, \sin i.$
5. $\text{cotan}\, p = \cos d \,.\, \sec a \,.\, \cot i + \sin d \,.\, \tan a.$
6. $\text{cotan}\, p = \cos l \,.\, \sec\lambda \,.\, \cot i - \sin l \,.\, \tan\lambda.$
7. $\cos a \,.\, \cos d = \cos l \,.\, \cos\lambda.$
8. $\sin p \,.\, \cos d = \sin i \,.\, \cos\lambda.$
9. $\sin p \,.\, \cos\lambda = \sin i \,.\, \cos d.$

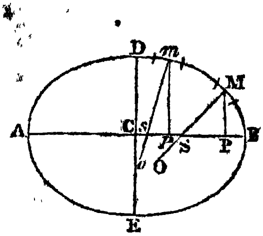
$$\left.\begin{array}{ll}10. & \tan a = \tan\lambda \,.\, \cos i. \\ 11. & \cos\lambda = \cos a \,.\, \cos d.\end{array}\right\} \text{when } l = 0, \text{ as is always the case with the sun.}$$

The

The investigation of these equations, which is omitted for the sake of brevity, depends on the resolution of the spherical triangle whose angles are the poles of the ecliptic and equator, and the given star, or luminary.

PROBLEM XIII.

To determine the Ratio of the Earth's Axes, and their Actual Magnitude, from the Measure of a Degree or Smaller Portion of a Meridian in Two Given Latitudes; the earth being supposed a spheroid generated by the rotation of an ellipse upon its minor axis.



Let ADBE represent a meridian of the earth, DE its minor axis, AB a diameter of the equator, M, m, arcs of the same number of degrees, or the same parts of a degree, of which the lengths are measured, and which are so small, compared with the magnitude of the earth, that they may be considered as coinciding with arcs of the osculatory circles at their respective middle points; let MO, mo, the radii of curvature of those middle points, be $=$ R and r respectively; MP, mp, ordinates perpendicular to AB: suppose further $\text{CD} = c$; $\text{CB} = d$; $d^2 - c^2 = e^2$; $\text{CP} = x$; $\text{C}p = u$; the radius or sine total $= 1$; the known angle ESM, or the latitude of the middle point M, $=$ L; the known angle Bsm, or the latitude of the point m, $= l$; the measured lengths of the arcs M and m being denoted by those letters respectively.

Now the similar sectors whose arcs are M, m, and radii of curvature R, r, give $\text{R} : r :: \text{M} : m$; and consequently $\text{R}m = r\text{M}$. The central equation to the ellipse investigated at p. 29 of this volume gives $\text{PM} = \frac{c}{d}\sqrt{(d^2 - x^2)}$; $pm = \frac{c}{d}\sqrt{(d^2 - u^2)}$; also $\text{SP} = \frac{e^2 x}{d^2}$; $sp = \frac{e^2 u}{d^2}$ (by th. 17 Ellipse). And the method of finding the radius of curvature (Flux. art. 74, 75), applied to the central equations above, gives

$\text{R} = \frac{(d^4 - e^2x^2)^{\frac{3}{2}}}{c^4 d}$; and $r = \frac{(d^4 - e^2u^2)^{\frac{3}{2}}}{c^4 d}$. On the other hand, the triangle SPM gives $\text{SP} : \text{PM} :: \cos \text{L} : \sin \text{L}$; that is, $\frac{e^2 x}{d^2} : \frac{c}{d}\sqrt{(d^2 - x^2)} :: \cos \text{L} : \sin \text{L}$; whence $x^2 = \frac{d^4 \cos^2 \text{L}}{d^2 - e^2 \sin^2 \text{L}}$. And from a like process there results, $u^2 = \frac{d^4 \cos^2 l}{d^2 - e^2 \sin^2 l}$. Substituting in the equation $\text{R}m = r\text{M}$, for R, and r their values,

values; for x^2 and u^2 their values just found, and observing that $\sin^2 L + \cos^2 L = 1$, and $\sin^2 l + \cos^2 l = 1$, we shall find

$$\frac{m}{(d^2 - e^2 \sin^2 L)^{\frac{3}{2}}} = \frac{M}{(d^2 - e^2 \sin^2 l)^{\frac{3}{2}}},$$

$$\text{or } m(d^2 - e^2 \sin^2 l)^{\frac{3}{2}} = M(d^2 - e^2 \sin^2 L)^{\frac{3}{2}},$$

$$\text{or } m^{\frac{2}{3}}(d^2 - e^2 \sin^2 l) = M^{\frac{2}{3}}(d^2 - e^2 \sin^2 L).$$

From this there arises $e^2 = d^2 - c^2$ (by hyp.) $=$ $\dfrac{d^2 (M^{\frac{2}{3}} - m^{\frac{2}{3}})}{M^{\frac{2}{3}} \sin^2 L - m^{\frac{2}{3}} \sin^2 l}$. But $\dfrac{c^2}{d^2} = 1 - \dfrac{d^2 - c^2}{d^2}$;

and consequently the reciprocal of this fraction, or

$$\frac{d^2}{c^2} = \frac{M^{\frac{2}{3}} \sin^2 L - m^{\frac{2}{3}} \sin^2 l}{m^{\frac{2}{3}} \cos^2 l - M^{\frac{2}{3}} \cos^2 L} = \frac{(M^{\frac{1}{3}} \sin L + m^{\frac{1}{3}} \sin l) \cdot (M^{\frac{1}{3}} \sin L - m^{\frac{1}{3}} \sin l)}{(m^{\frac{1}{3}} \cos l + M^{\frac{1}{3}} \cos L) \cdot (m^{\frac{1}{3}} \cos l - M^{\frac{1}{3}} \cos L)}.$$

Whence, by extracting the root, there results finally

$$\frac{d}{c} = \sqrt{\frac{(M^{\frac{1}{3}} \sin L + m^{\frac{1}{3}} \sin l) \cdot (M^{\frac{1}{3}} \sin L - m^{\frac{1}{3}} \sin l)}{(m^{\frac{1}{3}} \cos l + M^{\frac{1}{3}} \cos L) \cdot (m^{\frac{1}{3}} \cos l - M^{\frac{1}{3}} \cos L)}}.$$

This expression, which is simple and symmetrical, has been obtained without any developement into series, without any omission of terms on the supposition that they are indefinitely small, or any possible deviation from correctness, except what may arise from the want of coincidence of the circles of curvature at the middle points of the arcs measured, with the arcs themselves; and this source of error may be diminished at pleasure, by diminishing the magnitude of the arcs measured: though it must be acknowledged that such a procedure may give rise to errors in the practice, which may more than counterbalance the small one to which we have just adverted.

Cor. Knowing the number of degrees, or the parts of degrees, in the measured arcs M, m, and their lengths, which are here regarded as the lengths of arcs to the circle which have R, r, for radii, those radii evidently become known in magnitude. At the same time there are given the algebraic values of R and r: thus, taking R for example, and exterminating e^2 and x^2, there results $R = \dfrac{d^4}{c(d^2 - (d^2 - c^2) \sin^2 L)^{\frac{3}{2}}}$. Therefore, by putting in this equation, the known ratio of d to c, there will remain only one unknown quantity d or c, which may of course be easily determined by the reduction of the last equation; and thus all the dimensions of the terrestrial spheroid will become known.

General

General Scholium and Remarks.

1. The value $\frac{d}{c} - 1, = \frac{d-c}{c}$, is called the *compression* of the terrestrial spheroid, and it manifestly becomes known when the ratio $\frac{d}{c}$ is determined. But the measurements of philosophers, however carefully conducted, furnish resulting compressions, in which the discrepancies are much greater than might be wished. General Roy has recorded several of these in the Phil. Trans. vol. 77, and later measurers have deduced others. Thus, the degree measured at the equator by Bouguer, compared with that of France measured by Mechain and Delambre, gives for the compression $\frac{1}{334}$, also $d = 3271208$ toises, $c = 3261443$ toises, $d - c = 9765$ toises. General Roy's sixth spheroid, from the degrees at the equator and in latitude 45°, gives $\frac{1}{309 \cdot 3}$. Mr. Dalby makes $d =$ 3489932 fathoms, $c = 3473656$. Col. Mudge $d = 3491420$, $c = 3468007$, or 7935 and 7882 miles. The degree measured at Quito, compared with that measured in Lapland by Swanberg, gives compression $= \frac{1}{309 \cdot 4}$. Swanberg's observations, compared with Bouguer's, give $\frac{1}{329 \cdot 25}$. Swanberg's compared with the degree of Delambre and Mechain $\frac{1}{307 \cdot 4}$. Compared with Major Lambton's degree $\frac{1}{307 \cdot 17}$. A minimum of errors in Lapland, France, and Peru gives $\frac{1}{323 \cdot 4}$. Laplace, from the lunar motions, finds compression $= \frac{1}{314}$. From the theory of gravity as applied to the latest observations of Burg, Maskelyne, &c, $\frac{1}{309 \cdot 05}$. From the variation of the pendulum in different latitudes $\frac{1}{335 \cdot 78}$. Dr. Robison, assuming the variation of gravity at $\frac{1}{180}$, makes the compression $\frac{1}{319}$. Others give results varying from $\frac{1}{178 \cdot 4}$ to $\frac{1}{577}$: but far the greater number of observations differ but little from $\frac{1}{304}$, which the computation from the phenomena of the precession of the equinoxes and the nutation of the earth's axis gives for the maximum limit of the compression.

2. From the various results of careful admeasurements it happens, as Gen. Roy has remarked, "that philosophers are

 not

not yet agreed in opinion with regard to the exact figure of the earth; some contending that it has no regular figure, that is, not such as would be generated by the revolution of a curve around its axis. Others have supposed it to be an ellipsoid; regular, if both polar sides should have the same degree of flatness; but irregular if one should be flatter than the other. And lastly, some suppose it to be a spheroid differing from the ellipsoid, but yet such as would be formed by the revolution of a curve around its axis." According to the theory of gravity, however, the earth must of necessity have its axes approaching nearly to either the ratio of 1 to 680 or of 303 to 304; and as the former ratio obviously does not obtain, the figure of the earth *must* be such as to correspond nearly with the latter ratio.

3. Besides the method above described, others have been proposed for determining the figure of the earth by measurement. Thus, that figure might be ascertained by the measurement of a degree in two parallels of latitude; but not so accurately as by meridional arcs, 1st. Because, when the distance of the two stations, in the same parallel is measured, the celestial arc is not that of a parallel circle, but is nearly the arc of a great circle, and always exceeds the arc that corresponds truly with the terrestrial arc. 2dly. The interval of the meridian's passing through the two stations must be determined by a time-keeper, a very small error in the going of which will produce a very considerable error in the computation. Other methods which have been proposed, are, by comparing a degree of the meridian in any latitude, with a degree of the curve perpendicular to the meridian in the same latitude; by comparing the measures of degrees of the curve perpendicular to the meridian in different latitudes; and by comparing an arc of a meridian with an arc of the parallel of latitude that crosses it. The theorems connected with these and some other methods are investigated by Professor Playfair in the Edinburgh Transactions, vol. v, to which, together with the books mentioned at the end of the 1st section of this chapter, the reader is referred for much useful information on this highly interesting subject.

Having thus solved the chief problems connected with Trigonometrical Surveying, the student is now presented with the following examples by way of exercise.

Ex. 1. The angle subtended by two distant objects at a third object is 66°30′39″; one of those objects appeared under an elevation of 25′47″, the other under a depression of 1″. Required the reduced horizontal angle. Ans. 66°30′37″.

Ex. 2.

Ex. 2. Going along a straight and horizontal road which passed by a tower, I wished to find its height, and for this purpose measured two equal distances each of 84 feet, and at the extremities of those distances took three angles of elevation of the top of the tower, viz, 36° 50′, 21° 24′, and 14°. What is the height of the tower? Ans. 53·96 feet.

Ex. 3. Investigate General Roy's rule for the spherical excess, given in the scholium to prob. 8.

Ex. 4. The three sides of a triangle measured on the earth's surface (and reduced to the level of the sea) are 17, 18, and 10 miles: what is the spherical excess?

Ex. 5. The base and perpendicular of another triangle are 24 and 15 miles. Required the spherical excess.

Ex. 6. In a triangle two sides are 18 and 23 miles, and they include an angle of 58°24′36″. What is the spherical excess?

Ex. 7. The length of a base measured at an elevation of 58 feet above the level of the sea is 34286 feet: required the length when reduced to that level.

Ex. 8. Given the latitude of a place 48°51′N, the sun's declination 18°30′N, and the sun's altitude at $10^h 11^m 26^s$ AM, 52°35′; to find the angle that the vertical on which the sun is, makes with the meridian.

Ex. 9. When the sun's longitude is 29°13′43″, what is his right ascension? The obliquity of the ecliptic being 23°27′40″.

Ex. 10. Required the longitude of the sun, when his right ascension and declination are 32°46′52″$\frac{1}{2}$, and 13°13′27′ N respectively. See the theorems in the scholium to prob. 12.

Ex. 11. The right ascension of the star α Ursæ majoris is 162°50′34″, and the declination 62°50′N: what are the longitude and latitude? The obliquity of the ecliptic being as above.

Ex. 12. Given the measure of a degree on the meridian in N. lat. 49°3′, 60833 fathoms, and of another in N lat. 12°32′, 60494 fathoms: to find the ratio of the earth's axes.

Ex. 13. Demonstrate that, if the earth's figure be that of an oblate spheroid, a degree of the earth's equator is the first of two mean proportionals between the last and first degrees of latitude.

Ex. 14. Demonstrate that the degrees of the terrestrial meridian, in receding from the equator towards the poles, are

 increased

increased very nearly in the duplicate ratio of the sine of the latitude.

Ex. 15. If p be the measure of a degree of a great circle perpendicular to a meridian at a certain point, m that of the corresponding degree on the meridian itself, and d the length of a degree on an oblique arc, that arc making an angle a with the meridian, then is $d = \frac{pm}{p + (m - p) \sin^2 a}$. Required a demonstration of this theorem.

CHAPTER VI.

PRINCIPLES OF POLYGONOMETRY.

The theorems and problems in Polygonometry bear an intimate connection and close analogy to those in plane trigonometry; and are in great measure deducible from the same common principles. Each comprises three general cases.

1. A triangle is determined by means of two sides and an angle; or, which amounts to the same, by its sides except one, and its angles except two. In like manner, any rectilinear polygon is determinable when all its sides except one, and all its angles except two, are known.

2. A triangle is determined by one side and two angles; that is, by its sides except two, and all its angles. So likewise, any rectilinear figure is determinable, when all its sides except two, and all its angles, are known.

3. A triangle is determinable by its three sides; that is, when all its sides are known, and all its angles, but three. In like manner, any rectilinear figure is determinable by means of all its sides, and all its angles except three.

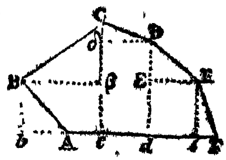
In each of these cases, the three unknown quantities may be determined by means of three independent equations; the manner of deducing which may be easily explained, after the following theorems are duly understood.

THEOREM I.

In Any Polygon, any One Side is Equal to the Sum of all the Rectangles of Each of the Other Sides drawn into the Cosine of the Angle made by that Side and the Proposed Side*.

* This theorem and the following one, were announced by Mr. Lexel of Petersburg, in Phil. Trans. vol. 65, p. 282: but they were first demonstrated by Dr. Hutton, in Phil. Trans. vol. 66, pa. 600.

Let



Let ABCDEF be a polygon: then will AF = AB . cos A + BC . cos CB ^ FA + CD . cos CD ^ AF + DE . cos DE ^ AF + EF . cos EF ^ AF*.

For, drawing lines from the several angles, respectively parallel and perpendicular to AF; it will be

$$\begin{aligned} Ab &= AB . \cos BAF, \\ bc &= B\beta = BC . \cos CB\beta = BC . \cos CB \wedge AF, \\ cd &= \delta D = CD . \cos CD\delta = CD . \cos CD \wedge AF, \\ de &= \epsilon E = DE . \cos DE\epsilon = DE . \cos DE \wedge AF, \\ eF &= \dots . EF . \cos EFe = EF . \cos EF \wedge AF. \end{aligned}$$

But AF = $bc + cd + de + e$F − Ab; and Ab, as expressed above, is in effect subtractive, because the cosine of the obtuse angle BAF is negative. Consequently,
AF = Ac + cd + de + eF = AB . cos BAF + BC . cos CB ^ AF + &c, as in the proposition. A like demonstration will apply, *mutatis mutandis*, to any other polygon.

Cor. When the sides of the polygon are reduced to three, this theorem becomes the same as the fundamental theorem in chap. ii, from which the whole doctrine of Plane Trigonometry is made to flow.

THEOREM II.

The Perpendicular let fall from the Highest Point or Summit of a Polygon, upon the Opposite Side or Base, is Equal to the Sum of the Products of the Sides Comprised between that Summit and the Base, into the Sines of their Respective Inclinations to that Base.

Thus, in the preceding figure, Cc = CB . sin CB ^ FA + BA . sin A; or Cc = CD . sin CD ^ AF + DE . sin DE ^ AF + EF . sin F. This is evident from an inspection of the figure.

Cor. 1. In like manner Dd = DE . sin DE ^ AF + EF . sin F, or Dd = CB . sin CB ^ FA + BA . sin A − CD . sin CD ^ AF.

Cor. 2. Hence, the sum of the products of each side, into the sine of the sum of the *exterior* angles, (or into the sine of the sum of the supplements of the interior angles), comprised between those sides and a determinate side, is = + perp. − perp. or = 0. That is to say, in the preceding figure,

$$AB . \sin A + BC . \sin (A + B) + CD . \sin (A + B + C) + DF . \sin (A + B + C + D) + EF . \sin (A + B + C + D + E) = 0.$$

* When a caret is put between two letters or pairs of letters denoting lines, the expression altogether denotes the angle which would be made by those two lines if they were produced till they met: thus CB ^ FA denotes the inclination of the line CB to FA.

Here

Here it is to be observed, that the sines of angles greater than $180°$ are negative (ch. ii equa. VII).

Cor. 3. Hence again, by putting for sin (A + B), sin (A + B + C), their values sin A . cos B + sin B . cos A, sin A . cos (B + C) + sin (B + C) . cos A, &c (ch. ii equa. V), and recollecting that tang $= \frac{\sin}{\cos}$ (ch. ii p. 55), we shall have,

$$\sin A . (AB + BC . \cos B + CD . \cos(B + C) + DE . \cos(B + C + D) + \&c)$$
$$+ \cos A . (BC . \sin B + CD . \sin(B + C) + DE . \cos(B + C + D) + \&c) = 0;$$

and thence finally, tan $180° - A$, or tan BAF $=$

$$\frac{BC . \sin B + CD . \sin(B + C) + DE . \sin(B + C + D) + EF . \sin(B + C + D + E)}{AB + BC . \cos B + CD . \cos(B + C) + DE . \cos(B + C + D) + EF . \cos(B + C + D + E)}.$$

A similar expression will manifestly apply to any polygon; and when the number of sides exceeds four, it is highly useful in practice.

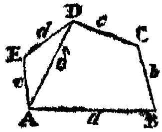
Cor. 4. In a triangle ABC, where the sides AB, BC, and the angle ABC, or its supplement B, are known, we have

$$\tan CAB = \frac{BC . \sin B}{AB + BC . \cos B} \ . \ . \ . \ . \tan BCA = \frac{AB . \sin B}{BC + AB . \cos B};$$

in both which expressions, the second term of the denominator will become subtractive whenever the angle ABC is acute, or B obtuse.

THEOREM III.

The Square of Any Side of a Polygon, is Equal to the Sum of the Squares of All the Other Sides, Minus Twice the Sum of the Products of All the Other Sides Multiplied two and two, and by the Cosines of the Angles they Include.



For the sake of brevity, let the sides be represented by the small letters which stand against them in the annexed figure: then, from theor. 1, we shall have the subjoined equations, viz.

$$a = b . \cos a^\wedge b + c . \cos a^\wedge c + \delta . \cos a^\wedge \delta,$$
$$b = a . \cos a^\wedge b + c . \cos b^\wedge c + \delta . \cos b^\wedge \delta,$$
$$c = a . \cos a^\wedge c + b . \cos b^\wedge c + \delta . \cos c^\wedge \delta,$$
$$\delta = a . \cos a^\wedge d + b . \cos b^\wedge d + c . \cos c^\wedge \delta,$$

Multiplying the first of these equations by a, the second by b, the third by c, the fourth by δ; subtracting the three latter products from the first, and transposing b^2, c^2, δ^2, there will result

$$a^2 = b^2 + c^2 + \delta^2 - 2(bc . \cos b^\wedge c + b\delta . \cos b^\wedge \delta + c\delta . \cos c^\wedge \delta).$$

In like manner,

$$c^2 = a^2 + b^2 + \delta^2 - 2(ab . \cos a^\wedge b + a\delta . \cos a^\wedge \delta + b\delta . \cos b^\wedge \delta).$$

&c. &c.

Or,

Or, since $b \wedge c = \text{C}$, $b \wedge \delta = \text{C} + \text{D} - 180^\circ$, $c \wedge \delta = \text{D}$, we have

$$a^2 = b^2 + c^2 + \delta^2 - 2(bc \,.\, \cos \text{C} - b\delta \,.\, \cos(\text{C}+\text{D}) + c\delta \,.\, \cos \text{D}),$$

$$c^2 = a^2 + b^2 + \delta^2 - 2(ab \,.\, \cos \text{B} - b\delta \,.\, \cos(\text{A}+\text{B}) + a\delta \,.\, \cos \text{A}),$$

&c. &c.

The same method applied to the pentagon ABCDE, will give

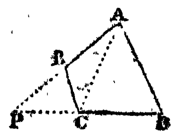
$$a^2 = b^2 + c^2 + d^2 + e^2 - 2 \left\{ \begin{array}{l} bc \,.\, \cos \text{C} - bd \,.\, \cos(\text{C}+\text{D}) + be \,.\, \cos(\text{C}+\text{D}+\text{E}) \\ + cd \,.\, \cos \text{D} - ce \,.\, \cos(\text{D}+\text{E}) + de \,.\, \cos \text{E} \end{array} \right\}.$$

And a like process is obviously applicable to any number of sides; whence the truth of the theorem is manifest.

Cor. The property of a plane triangle expressed in equ. 1. ch. ii, is only a particular case of this general theorem.

THEOREM IV.

Twice the Surface of Any Polygon, is Equal to the Sum of the Rectangles of its Sides, except one, taken two and two, by the Sines of the Sums of the *Exterior** Angles Contained by those Sides.



1. For a trapezium, or polygon of four sides. Let two of the sides AB, DC, be produced till they meet at P. Then the trapezium ABCD is manifestly equal to the difference between the triangles PAD and PBC. But twice the surface of the triangle PAD is (Mens. of Planes pr. 2 rule 2) AP . PD . sin P = (AB + BP) . (DC + CP) . sin P; and twice the surface of the triangle PBC is = BP . PC . sin P: therefore their difference, or twice the area of the trapezium, is = (AB . DC + AB . CP + DC . BP) . sin P. Now, in $\triangle$ PBC,

$$\sin \text{P} : \sin \text{B} :: \text{BC} : \text{PC}, \text{ whence } \text{PC} = \frac{\text{BC} \,.\, \sin \text{B}}{\sin \text{P}},$$

$$\sin \text{P} : \sin \text{C} :: \text{BC} : \text{PB}, \text{ whence } \text{PB} = \frac{\text{BC} \,.\, \sin \text{C}}{\sin \text{P}}.$$

Substituting these values of PB, PC, for them in the above equation, and observing that sin P = sin (PBC + PCB) = sin sum of *exterior* angles B and C, there results at length,

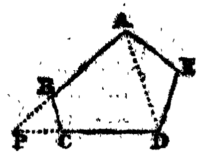
$$\left.\begin{array}{r} \text{Twice surface} \\ \text{of trapezium.} \end{array}\right\} = \left\{ \begin{array}{l} \text{AB} \,.\, \text{BC} \,.\, \sin \text{B} \\ + \text{AB} \,.\, \text{DC} \,.\, \sin(\text{B}+\text{C}) \\ + \text{BC} \,.\, \text{DC} \,.\, \sin \text{C}. \end{array} \right.$$

Cor. Since AB . BC . sin B = twice triangle ABC, it follows that twice triangle ACD is equal to the remaining two terms, viz,

$$\text{twice area ACD} = \left\{ \begin{array}{l} \text{AB} \,.\, \text{DC} \,.\, \sin(\text{B}+\text{C}) \\ + \text{BC} \,.\, \text{DC} \,.\, \sin \text{C}. \end{array} \right.$$

* The exterior angles here meant, are those formed by producing the sides in the same manner as in th. 20 Geometry, and in cors. 1, 2, th. 2, of this chap.

2. For



2. For a pentagon, as ABCDE. Its area is obviously equal to the sum of the areas of the trapezium ABCD, and of the triangle ADE. Let the sides AB, DC, as before, meet when produced at P. Then, from the above, we have

$$\left.\begin{array}{r}\text{Twice area of}\\ \text{the trapezium}\\ \text{ABCD}\end{array}\right\} = \left\{\begin{array}{l}\text{AB . BC . sin B}\\ +\text{ AB . DC . sin (B + C)}\\ +\text{ BC . DC . sin C.}\end{array}\right.$$

And, by the preceding corollary,

$$\left.\begin{array}{r}\text{Twice triangle}\\ \text{DAE}\end{array}\right\} = \left\{\begin{array}{l}\text{AP . DE . sin (P + D) or sin (B + C + D)}\\ +\text{ DP . DE . sin D.}\end{array}\right.$$

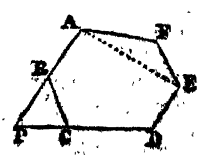
$$\left.\begin{array}{r}\text{That is, twice}\\ \text{triangle DAE}\end{array}\right\} = \left\{\begin{array}{l}\text{AB . DE . sin (B + C + D)}\\ +\text{ DC . DE . sin D}\\ +\text{ BP . DE . sin (B + C + D)}\\ +\text{ CP . DE . sin D.}\end{array}\right.$$

Now, $\text{BP} = \frac{\text{BC . sin C}}{\sin(\text{B + C})}$, and $\text{CP} = \frac{\text{BC . sin B}}{\sin(\text{B + C})}$: therefore the last two terms become $\frac{\text{BC . DE . sin C . sin (B + C + D)}}{\sin(\text{B + C})} + \frac{\text{BC . DE . sin B . sin D}}{\sin(\text{B + C})}$ $= \text{BC . DE} . \frac{\text{sin B . sin D + sin C . sin (B + C + D)}}{\sin(\text{B + C})}$: and this expression, by means of the formula for 4 arcs (art. 30 ch. iii), becomes BC . DE . sin (C + D). Hence, collecting the terms, and arranging them in the order of the sides, they become

$$\left.\begin{array}{r}\text{Twice the area}\\ \text{of the penta-}\\ \text{gon ABCDE}\end{array}\right\} = \left\{\begin{array}{l}\text{AB . BC . sin B}\\ +\text{ AB . DC . sin (B + C)}\\ +\text{ AB . DE . sin (B + C + D)}\\ +\text{ BC . DC . sin C}\\ +\text{ BC . DE . sin (C + D)}\\ +\text{ DC . DE . sin D.}\end{array}\right.$$

Cor. Taking away from this expression the 1st, 2d, and 4th terms, which together make double the trapezium ABCD, there will remain

$$\left.\begin{array}{r}\text{Twice area of}\\ \text{the triangle}\\ \text{DAE.}\end{array}\right\} = \left\{\begin{array}{l}\text{AB . DE . sin (B + C + D)}\\ +\text{ BC . DE . sin (C + D)}\\ +\text{ DC . DE . sin D.}\end{array}\right.$$



3. For a hexagon, as ABCDEF. The double area will be found, by supposing it divided into the pentagon ABCDE, and the triangle AEF. For, by the last rule, and its corollary, we have,

Twice

$$\left.\begin{array}{r}\text{Twice area of}\\ \text{the pentagon}\\ \text{ABCDE}\end{array}\right\}=\left\{\begin{array}{l}\quad AB . BC . \sin B\\ + AB . CD . \sin (B+C)\\ + AB . DE . \sin (B+C+D)\\ + BC . CD . \sin C\\ + BC . DE . \sin (C+D)\\ + CD . DE . \sin D.\end{array}\right.$$

$$\left.\begin{array}{r}\text{Twice area of}\\ \text{the triangle}\\ \text{AEF}\end{array}\right\}=\left\{\begin{array}{l}\quad AP . EF . \sin (B+C+D+E)\\ + DP . EF . \sin (D+E)\\ + DE . EF . \sin E.\end{array}\right.$$

$$\left.\begin{array}{r}\text{Or, twice area of}\\ \text{the triangle}\\ \text{AEF.}\end{array}\right\}=\left\{\begin{array}{l}\quad AB . EF . \sin (B+C+D+E)\\ + DC . EF . \sin (D+E)\\ + DE . EF . \sin E\\ + BP . EF . \sin (B+C+D+E)\\ + CP . EF . \sin (D+E).\end{array}\right.$$

Now, writing for BP, CP, their respective values, $\frac{BC . \sin C}{\sin (B+C)}$ and $\frac{BC . \sin B}{\sin (B+C)}$, the sum of the last two expressions, in the double areas of AEF, will become

$$BC . EF . \frac{\sin C . \sin (B+C+D+E) + \sin B . \sin (D+E)}{\sin (B+C)}:$$

and this, by means of the formula for 5 arcs (art. 30 ch. iii) becomes BC . EF . sin (C + D + E). Hence, collecting and properly arranging the several terms as before, we shall obtain

$$\left.\begin{array}{r}\text{Twice the area}\\ \text{of the hexa-}\\ \text{gon ABCDEF}\end{array}\right\}=\left\{\begin{array}{l}\quad AB . BC . \sin B\\ + AB . CD . \sin (B+C)\\ + AB . DE . \sin (B+C+D)\\ + AB . EF . \sin (B+C+D+E)\\ + BC . CD . \sin C\\ + BC . DE . \sin (C+D)\\ + BC . EF . \sin (C+D+E)\\ + CD . DE . \sin D\\ + CD . EF . \sin (D+E)\\ + DE . EF . \sin E.\end{array}\right.$$

4. In a similar manner may the area of a heptagon be determined, by finding the sum of the areas of the hexagon and the adjacent triangle; and thence the area of the octagon, nonagon, or of any other polygon, may be inferred; the law of continuation being sufficiently obvious from what is done above, and the number of terms $= \frac{n-1}{1} . \frac{n-2}{2}$, when the number of sides of the polygon is n: for the number of terms is evidently the same as the number of ways in which $n-1$ quantities can be taken, two and two; that is, (by the nature of Permutations) $= \frac{n-1}{1} . \frac{n-2}{2}$.

Scholium.

Scholium.

This curious theorem was first investigated by *Simon Lhuillier*, and published in 1789. Its principal advantage over the common method for finding the areas of irregular polygons is, that in this method there is no occasion to construct the figures, and of course the errors that may arise from such constructions are avoided.

In the application of the theorem to practical purposes, the expressions above become simplified by dividing any proposed polygon into two parts by a diagonal, and computing the surface of each part separately.

Thus, by dividing the trapezium ABCD into two triangles, by the diagonal AC, we shall have

$$\left.\begin{array}{c}\text{Twice area}\\ \text{trapezium}\end{array}\right\} = \left\{\begin{array}{l}AB \cdot BC \cdot \sin B\\ + CD \cdot AD \cdot \sin D.\end{array}\right.$$

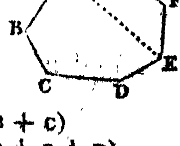
The pentagon ABCDE may be divided into the trapezium ABCD, and the triangle ADE, whence

$$\left.\begin{array}{c}\text{Twice area of}\\ \text{pentagon}\end{array}\right\} = \left\{\begin{array}{l}AB \cdot BC \cdot \sin B\\ + AB \cdot DC \cdot \sin (B + C)\\ + BC \cdot DC \cdot \sin C\\ + DE \cdot AE \cdot \sin E.\end{array}\right.$$

Thus again, the hexagon may be divided into two trapeziums, by a diagonal drawn from A to D, which is to be the line excepted in the theorem; then will

$$\left.\begin{array}{c}\text{Twice area of}\\ \text{hexagon}\end{array}\right\} = \left\{\begin{array}{l}AB \cdot BC \cdot \sin B\\ + AB \cdot DC \cdot \sin (B + C)\\ + BC \cdot DC \cdot \sin C\\ + DE \cdot EF \cdot \sin E\\ + DE \cdot AF \cdot \sin (E + F)\\ + EF \cdot AF \cdot \sin F.\end{array}\right.$$

And lastly, the heptagon may be divided into a pentagon and a trapezium, the diagonal, as before, being the excepted line: so will the double area be expressed by 9 instead of 15 products, thus:



$$\left.\begin{array}{c}\text{Twice area of}\\ \text{heptagon}\end{array}\right\} = \left\{\begin{array}{l}AB \cdot BC \cdot \sin B\\ + AB \cdot CD \cdot \sin (B + C)\\ + AB \cdot DE \cdot \sin (B + C + D)\\ + BC \cdot CD \cdot \sin C\\ + BC \cdot DE \cdot \sin (C + D)\\ + CD \cdot DE \cdot \sin D\\ + EF \cdot FG \cdot \sin F\\ + EF \cdot GA \cdot \sin (F + G)\\ + FG \cdot GA \cdot \sin G.\end{array}\right.$$

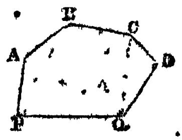
The same method may obviously be extended to other polygons, with great ease and simplicity.

It

It often happens, however, that only one side of a polygon can be measured, and the distant angles be determined by intersection; in this case the area may be found, independent of construction, by the following problem.

PROBLEM I.

Given the Length of One of the Sides of a Polygon, and the Angles made at its two extremities by that Side and Lines drawn to all the Other Angles of the Polygon; to find an Expression for the Surface of that Polygon.



Here we suppose known PQ; also $APQ = a'$, $BPQ = b'$, $CPQ = c'$, $DPQ = d'$; $AQP = a''$, $BQP = b''$, $CQP = c''$, $DQP = d''$. Now, $\sin PAQ = \sin(a' + a'')$; $\sin PBQ = \sin(b' + b'')$.

Therefore, $\sin(a' + a'') : PQ :: \sin a'' : PA = \dfrac{\sin a''}{\sin(a' + a'')} PQ$.

And, . . . $\sin(b' + b'') . PQ :: \sin b'' . PB = \dfrac{\sin b''}{\sin(b' + b'')} PQ$.

But, triangle $APB = AP . PB . \frac{1}{2} \sin APB = \frac{1}{2} AP . PB . \sin(a' - b')$

Hence, surface $\Delta\ APB = \frac{1}{2}PQ^2 . \dfrac{\sin a'' . \sin b'' . \sin(a' - b')}{\sin(a' + a'')\ \sin(b' + b'')}$.

In like manner, $\Delta\ BPC = \frac{1}{2}PQ^2 . \dfrac{\sin b''\ \sin c''\ \sin(b' - c')}{\sin(b' + b'')\ \sin(c' + c'')}$.

$$\Delta\ CPD = \tfrac{1}{2}PQ^2 . \frac{\sin c''\ \sin d''\ \sin(c' - d')}{\sin(c' + c'')\ \sin(d' + d'')}.$$

&c. &c. &c.

$\Delta\, DPQ = QP . PD . \frac{1}{2} \sin DPQ = PQ . \dfrac{\sin d''}{\sin(d' + d'')} . \frac{1}{2} PQ . \sin d' =$

$\frac{1}{2} PQ^2 . \dfrac{\sin d' . \sin d''}{\sin(d' + d'')}$. Consequently,

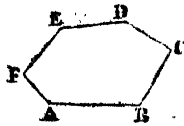
$$\text{Surface } PABCDQ = \tfrac{1}{2} PQ^2 . \begin{cases} \dfrac{\sin a''\ \sin b'' . \sin(a' - b')}{\sin(a' + a'')\ \sin(b' + b'')} \\ + \dfrac{\sin b''\ \sin c''\ \sin(b' - c')}{\sin(b' + b'') . \sin(c' + c'')} \\ + \dfrac{\sin c'' . \sin d''\ \sin(c' - d')}{\sin(c' + c'')\ \sin(d' + d'')} \\ + \dfrac{\sin d' . \sin d''}{\sin(d' + d'')} \end{cases}$$

The same method manifestly applies to polygons of *any* number of sides: and all the terms except the last are so perfectly symmetrical, while that last term is of so obvious a form, that there cannot be the least difficulty in extending the formula to any polygon whatever.

PROBLEM II.

Given, in a Polygon, All the Sides and Angles, except three; to find the Unknown Parts.

This problem may be divided into three general cases, as shown at the beginning of this chapter: but the analytical solution of all of them depends on the same principles; and these are analogous to those pursued in the analytical investigations of plane trigonometry. In polygonometry, as well as trigonometry, when three unknown quantities are to be found, it must be by means of three independent equations, involving the known and unknown parts. These equations may be deduced from either theorem 1, or 3, as may be most suited to the case in hand; and then the unknown parts may each be found by the usual rules of extermination.



For an example, let it be supposed that in an irregular hexagon ABCDEF, there are given all the sides except AB, BC, and all the angles except B; to determine those three quantities.

The angle B is evidently equal to $(2n-4)$ right angles $-$ (A + C + D + E + F); n being the number of sides, and the angles being here supposed the interior ones.

Let AB $= x$, BC $= y$: then, by th. 1,

$$x = y \cdot \cos B + DC \cdot \cos AB \,\hat{}\, CD + DE \cdot \cos AB \,\hat{}\, ED + EF \cdot \cos AB \,\hat{}\, EF + AF \cdot \cos AB \,\hat{}\, AF;$$

$$y = x \cdot \cos B + AF \cdot \cos BC \,\hat{}\, AF + FE \cdot \cos BC \,\hat{}\, FE + DE \cdot \cos BC \,\hat{}\, DE + DC \cdot \cos BC \,\hat{}\, CD.$$

In the first of the above equations, let the sum of all the terms after y . cos B, be denoted by c; and in the second the sum of all those which fall after x . cos B, by d; both sums being manifestly constituted of known terms: and let the known coefficients of x and y be m and n respectively. Then will the preceding equations become

$$x = ny + c \;.\;.\;.\; y = mx + d.$$

Substituting for y, in the first of the two latter equations, its value in the second, we obtain $x = mnx + nd + c$. Whence there will readily be found

$$x = \frac{nd + c}{1 - mn}, \text{ and } y = \frac{mc + d}{1 - mn}.$$

Thus AB and BC are determined. Like expressions will serve for the determination of any other two sides, whether contiguous or not: the coefficients of x and y being designated by different letters for that express purpose; which would have been otherwise unnecessary in the solution of the individual case proposed.

Remark.

Remark. Though the algebraic investigations commonly lead to results which are apparently simple, yet they are often, especially in polygons of many sides, inferior in practice to the methods suggested by subdividing the figures. The following examples are added for the purpose of explaining those methods: the operations however are merely indicated; the detail being omitted to save room.

EXAMPLES.

Ex. 1. In a hexagon ABCDEF, all the sides except AF, and all the angles except A and F, are known. Required the unknown parts. Suppose we have

	Ext. ang.	Whence	
AB = 1284			
BC = 1782	B = 32°	B + C	= 80°
CD = 2400	C = 48°	B + C + D	= 132°
DE = 2700	D = 52°	B + C + D + E	= 198°
EF = 2860	E = 66°	A + F	= 162°.

Then, by cor 3 th. 2, tan BAF =

$$\frac{\text{BC} \cdot \sin \text{B} + \text{CD} \cdot \sin(\text{B}+\text{C}) + \text{DE} \cdot \sin(\text{B}+\text{C}+\text{D}) + \text{EF} \cdot \sin(\text{B}+\text{C}+\text{D}+\text{E})}{\text{AB} + \text{BC} \cdot \cos \text{B} + \text{CD} \cdot \cos(\text{B}+\text{C}) + \text{DE} \cdot \cos(\text{B}+\text{C}+\text{D}) + \text{EF} \cdot \cos(\text{B}+\text{C}+\text{D}+\text{E})}$$

$$= \frac{\text{BC} \cdot \sin 32^\circ + \text{CD} \cdot \sin 80^\circ + \text{DE} \cdot \sin 132^\circ + \text{EF} \cdot \sin 198^\circ}{\text{AB} + \text{BC} \cdot \cos 32^\circ + \text{CD} \cdot \cos 80^\circ + \text{DE} \cdot \cos 132^\circ + \text{EF} \cdot \cos 198^\circ}$$

$$= \frac{\text{BC} \cdot \sin 32^\circ + \text{CD} \cdot \sin 80^\circ + \text{DE} \cdot \sin 48^\circ - \text{EF} \cdot \sin 18^\circ}{\text{AB} + \text{BC} \cdot \cos 32^\circ + \text{CD} \cdot \cos 80^\circ - \text{DE} \cdot \cos 48^\circ - \text{EF} \cdot \cos 18^\circ}.$$

Whence BAF is found 106°31′38″; and the other angle AFE = 91°28′22″. So that the exterior angles A and F are 73°28′22″, and 88°31′38″ respectively: all the exterior angles making 4 right angles, as they ought to do. Then, all the angles being known, the side AF is found by th. 1 = 4621·5.

If one of the angles had been a re-entering one, it would have made no other difference in the computation than what would arise from its being considered as subtractive.

Ex. 2. In a hexagon ABCDEF, all the sides except AF, and all the angles except C and D, are known: viz,

	Ex. Ang.
AB = 2400	
BC = 2700	A = 54°
CD = 3200	B = 62°
DE = 3500	E = 64°
EF = 3750	F = 72°

We shall have, by th. 2 cor 1,

$$\left.\begin{array}{l} \text{AB} \cdot \sin \text{A} \\ + \text{BC} \cdot \sin(\text{A}+\text{B}) \\ + \text{CD} \cdot \sin(\text{A}+\text{B}+\text{C}) \end{array}\right\} = \left\{\begin{array}{l} \text{DE} \cdot \sin(\text{E}+\text{F}) \\ + \text{EF} \cdot \sin \text{F}. \end{array}\right.$$

$$\text{Therefore, CD} \cdot \sin(116^\circ + \text{C}) = \left\{\begin{array}{l} - \text{AB} \cdot \sin 54^\circ \\ - \text{BC} \cdot \sin 116^\circ \\ + \text{DE} \cdot \sin 136^\circ \\ + \text{EF} \cdot \sin 72^\circ. \end{array}\right.$$

$$\text{Or, } 116^\circ + \text{C} = \left\{\begin{array}{l} 149^\circ 23' 26'', \\ + 33^\circ 36' 34''. \end{array}\right.$$

The

The second of these will give for C, a re-entering angle; the first will give exterior angle $C = 33°23'26''$, and then will $D = 14°36'34''$. Lastly,

$$AF = \left\{\begin{array}{l} -AB \cdot \cos 54° \\ +BC \cdot \cos 64° \\ +CD \cdot \cos 30°36'34'' \\ +DE \cdot \cos 44° \\ -EF \cdot \cos 72°. \end{array}\right\} = 3885{\cdot}905.$$

Ex. 3. In a hexagon ABCDEF, are known, all the sides except AF, and all the angles except B and E; to find the rest.

Given $AB = 1200$		Exterior angles $A = 64°$	
$BC = 1500$			
$CD = 1600$		$C = 72°$	
$DE = 1800$		$D = 75°$	
$EF = 2000$			
		$F = 84°$.	

Suppose the diagonal BE drawn, dividing the figure into two trapeziums. Then, in the trapezium BCDE, the sides except BE, and the angles except B and E, will be known; and these may be determined as in exam. 1. Again, in the trapezium ABEF, there will be known the sides except AF, and the angles except the adjacent ones B and E. Hence, first for BCDE: (cor. 3 th. 2).

$$\tan CBE = \frac{CD \cdot \sin C + DE \cdot \sin (C + D)}{BC + CD \cdot \cos C + DE \cdot \cos (C + D)} =$$

$$\frac{CD \cdot \sin 72° + DE \cdot \sin 147°}{BC + CD \cdot \cos 72° + DE \cdot \cos 147°} = \frac{CD \cdot \sin 72° + DE \cdot \sin 33°}{BC + CD \cdot \cos 72° - DE \cdot \cos 33°}.$$

Whence $CBE = 79°2'1''$; and therefore $DEB = 67°57'59''$.

$$\text{Then } EB = \left\{\begin{array}{l} BC \cdot \cos 79°\,2' \\ +CD \cdot \cos \;\; 7°\,2'\,1'' \\ +DE \cdot \cos 67°57'59'' \end{array}\right\} = 2548{\cdot}581.$$

Secondly, in the trapezium ABEF,

$AB \cdot \sin A + BE \cdot \sin (A + B) = EF \cdot \sin F$: whence

$$\sin (A + B) = \frac{EF \cdot \sin F - AB \cdot \sin B}{BE} = \sin \left\{\begin{array}{l} 20°55'54'', \\ 159°\,4'\,6''. \end{array}\right.$$

Taking the lower of these, to avoid re-entering angles, we have B (exterior ang.) $= 95°4'6''$; $ABE = 84°55'54''$; $FEB = 63°4'6''$: therefore $ABC = 163°57'55''$; and $FED = 131°2'5''$: and consequently the exterior angles at B and E are $16°2'5''$ and $48°57'55''$ respectively.

Lastly, $AF = -AB \cdot \cos A - BE \cdot \cos (A + B) - EF \cdot \cos F = -AB \cdot \cos 64° + BE \cdot \cos 20°55'54'' - EF \cdot \cos 84° = 1645{\cdot}292$.

Note. The preceding three examples comprehend all the varieties which can occur in Polygonometry, when all the sides except one, and all the angles but two, are known. The unknown angles may be about the unknown side; or they may be

be adjacent to each other, though distant from the unknown side; and they may be remote from each other, as well as from the unknown side.

Ex. 4. In a hexagon ABCDEF, are known all the angles, and all the sides except AF and CD: to find those sides.

Given AB = 2200		Ext. Ang. A =	96°
BC = 2400		B =	54°
		C =	20°
DE = 4800		D =	24°
EF = 5200		E =	18°
		F =	148°.

Here, reasoning from the principle of cor. th. 2, we have

$$\left.\begin{array}{r} AB \,.\, \sin\ 96^\circ \\ + BC \,.\, \sin 150^\circ \\ + CD \,.\, \sin 170^\circ \end{array}\right\} = \left\{\begin{array}{l} DE \,.\, \sin 166^\circ \\ + EF \,.\, \sin 148^\circ. \end{array}\right. \quad \text{or } \left.\begin{array}{r} AB \,.\, \sin 84^\circ \\ + BC \,.\, \sin 30^\circ \\ + CD \,.\, \sin 10^\circ \end{array}\right\} = \left\{\begin{array}{l} DE \,.\, \sin 14^\circ \\ + EF \,.\, \sin 32^\circ. \end{array}\right.$$

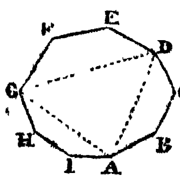
$$\begin{array}{c}\text{Whence}\\ CD =\end{array}\left\{\begin{array}{l} DE \,.\, \sin 14^\circ \,.\, \operatorname{cosec} 10^\circ - AB \,.\, \sin 84^\circ \,.\, \operatorname{cosec} 10^\circ \\ + EF \,.\, \sin 32^\circ\ \operatorname{cosec} 10^\circ - BC \,.\, \sin 30^\circ \,.\, \operatorname{cosec} 10^\circ \end{array}\right\} = 3045{\cdot}58.$$

$$\begin{array}{c}\text{And}\\ AF =\end{array}\left\{\begin{array}{l} DE \,.\, \sin 24^\circ \,.\, \operatorname{cosec} 10^\circ - CB \,.\, \sin 20^\circ \\ + EF \,.\, \sin 42^\circ \,.\, \operatorname{cosec} 10^\circ - BA \,.\, \sin 74^\circ \end{array}\right\} = 14374{\cdot}98.$$

Ex. 5. In the nonagon ABCDEFGHI, all the sides are known, and all the angles except A, D, G: it is required to find those angles.

Given AB = 2400	FG = 3800	Ext. Ang. B =	40°
BC = 2700	GH = 4000	C =	32°
CD = 2800	HI = 4200	E =	36°
DE = 3200	IA = 4500.	F =	45°
EF = 3500		H =	48°
		I =	50°.

Suppose diagonals drawn to join the unknown angles, and dividing the polygon into three trapeziums and a triangle; as in the marginal figure. Then,



1st. In the trapezium ABCD, where AD and the angles about it are unknown; we have (cor. 3 th. 2)

$$\tan BAD = \frac{BC \,.\, \sin B + CD \,.\, \sin (B + C)}{AB + BC \,.\, \cos B + CD \,.\, \cos (B + C)} = \frac{BC \,.\, \sin 40^\circ + CD \,.\, \sin 72^\circ}{AB + BC \,.\, \cos 40^\circ + CD \,.\, \cos 72^\circ}.$$

Whence BAD = 39° 30′42″, CDA = 32° 29′18″.

$$\text{And } AD = \left\{\begin{array}{l} \quad AB \,.\, \cos 39^\circ 30' 42'' \\ + BC \,.\, \cos\ \ 0\ 29\ 18 \\ + CD \,.\, \cos 32\ 29\ 18 \end{array}\right\} = 6913{\cdot}292.$$

2dly. In the quadrilateral DEFG, where DG and the angles about it are unknown; we have

$$\tan EDG = \frac{EF \,.\, \sin E + FG \,.\, \sin (E + F)}{DE + EF \,.\, \cos E + FG \,.\, \cos (E + F)} = \frac{EF \,.\, \sin 36^\circ + FG \,.\, \sin 81^\circ}{DE + EF \,.\, \cos 36^\circ + FG \,.\, \cos 81^\circ}.$$

Whence

Whence EDG $= 41°\ 14'\ 53''$, FGD $= 39°\ 45'\ 7''$.

$$\text{And } DG = \left\{\begin{array}{l} DE \,.\, \cos 41°14'53'' \\ + EF \,.\, \cos\ \ 5°14'53'' \\ + FG \,.\, \cos 39°45'\ 7'' \end{array}\right\} = 8812{\cdot}803.$$

3dly. In the trapezium GHIA, an exactly similar process gives HGA $= 50°\ 46'\ 53''$, IAG $= 47°\ 13'\ 7''$, and AG $= 9780{\cdot}591$.

4thly. In the triangle ADG, the three sides are now known, to find the angles: viz, DAG $= 60°\ 53'\ 26''$, AGD $= 43°\ 15'\ 54''$, ADG $= 75°\ 50'\ 40''$. Hence there results, lastly,

IAB $= 47°13'\ 7''+60°\ 53'\ 26''+39°\ 30\ 42''=147°\ 37'\ 15''$,
CDE $= 32°29'18''+70°\ 50'\ 40''+41°\ 14'\ 53''=149°\ 34'\ 51''$,
FGH $= 39°45'\ 7''+43°\ 15'\ 54''+50°\ 46'\ 53''=133°\ 47'\ 54''$.

Consequently, the required exterior angles are A $= 32°22'45''$, D $= 30°\ 25'\ 9''$, G $= 46°\ 12'\ 6''$.

Ex. 6. Required the area of the hexagon in ex. 1.

Ans. 16530191.

Ex. 7. In a quadrilateral ABCD, are given AB $= 24$, BC $= 30$, CD $= 34$; angle ABC $= 92°18'$, BCD $= 97°23'$. Required the side AD, and the area.

Ex. 8. In prob. 1, suppose PQ $= 2538$ links, and the angles as below; what is the area of the field ABCDQP?

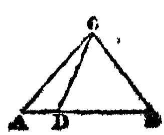
APQ$=89°14'$, BPQ$=68°11'$, CPQ$=36°24'$, DPQ$=\ 19°57'$,
AQP$=25°18'$, BQP$=69°24'$, CQP$=94°\ 6$, DQP$=121°18'$.

CHAPTER VII.

PROBLEMS RELATIVE TO THE DIVISION OF FIELDS OR OTHER SURFACES.

PROBLEM I.

To Divide a Triangle into Two Parts having a Given Ratio, $m : n$.

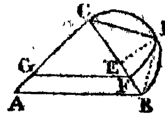


1st. By a line drawn from one angle of the triangle.

Make AD : AB : : $m : m + n$; draw CD. So shall ADC, BDC, be the parts required.

Here, evidently, AD $= \frac{m}{m+n}$ AB, DB $= \frac{n}{m+n}$ AB.

2dly. By a line parallel to one of the sides of the triangle.

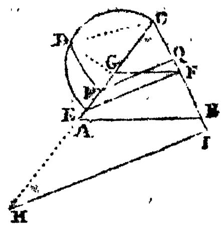


Let ABC be the given triangle, to be divided into two parts, in the ratio of m to n, by a line parallel to the base AB. Make CE to EB as m to n: erect ED perpendicularly to CB, till it meet the semicircle described on CB, as a diameter, in D. Make CF = CD: and draw through F, GF ∥ AB. So shall GF divide the triangle ABC in the given ratio.

For, CE : CB = $\frac{CD^2}{CE}$: : $CD^2(=CF^2) : CB^2$. But CE : EB : : m : n, or CE : CB : : m : $m+n$, by the construction: therefore, $CF^2 : CB^2 : : m : m+n$. And since Δ CGF : Δ CAB : : $CF^2 : CB^2$; it follows that CGF : CAB : : m : $m+n$, as required.

Computation. Since $CB^2 : CF^2 : : m+n : m$, therefore, $(m+n)CF^2 = m \cdot CB^2$; whence $CF\sqrt{(m+n)} = CB\sqrt{m}$, or $CF = CB\sqrt{\frac{m}{m+n}}$. In like manner, $CG = CA\sqrt{\frac{m}{m+n}}$.

3dly. By a line parallel to a given line.



Let HI be the line parallel to which a line is to be drawn, so as to divide the triangle ABC in the ratio of m to n.

By case 2d draw GF parallel to AB, so as to divide ABC in the given ratio. Through F draw FE parallel to HI. On CE as a diameter describe a semicircle; draw GD perp. to AC, to cut the semicircle in D. Make CP = CD: through P, parallel to EF, draw PQ, the line required.

The demonstration of this follows at once from case 2; because it is only to divide FCE, by a line parallel to FE, into two triangles having the ratio of FCE to FCG, that is, of CE to CG.

Computation. CG and CF being computed, as in case 1, the distances CH, CI being given, and CP being to CQ as CH to CI: the triangles CGF, GPQ, also having a common vertical angle, are to each other, as CG . CF to CQ . CP. These products therefore are equal; and since the factors of the former are known, the latter product is known. We have hence given the ratio of the two lines CP($=x$) to CQ($=y$) as CH to CI; say, as p to q; and their product = CF . CG, say $= ab$: to find x and y. Here we find $x = \sqrt{\frac{abp}{q}}$, $y = \sqrt{\frac{abq}{p}}$. That is,

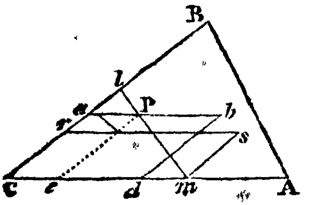
$$CP = \sqrt{\frac{CF \cdot CG \cdot CH}{CI}};\ CQ = \sqrt{\frac{CF \cdot CG \cdot CI}{CH}}.$$

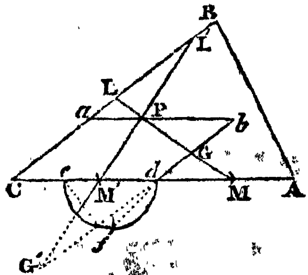
N. B. If the line of division were to be perpendicular to one of the sides, as to CA, the construction would be similar:

 CP

CP would be a geometrical mean between CA and $\frac{m}{m+n}$Cb, b being the foot of the perpendicular from B upon AC.

4thly. By a line drawn through a given point P.





By any of the former cases draw lm (fig. 1) to divide the triangle ABC, in the given ratio of m to n: bisect Cl in r, and through r and m let pass the sides of the rhomboid Crsm. Make Ca = Pe, which is given, because the point P is given in position: make Cd a fourth proportional to Ca, Cr, Cm; that is, make Ca : Cr : : Cm : Cd; and let a, and d, be two angles of the rhomboid Cabd, figs. 1 and 2. Pe, in figure 2, being drawn parallel to AC, describe on ed as a diameter the semicircle efd, on which set off ef = Ce = aP: then set off dM or dM′ on CA equal to df, and through P and M, P and M′, draw the lines LM, L′M′, either of which will divide the triangle in the given ratio.—The construction is given in 2 figs. merely to avoid complexness in the diagrams.

The limitations are obvious from the construction: for, the point L must fall between B and C, and the point M between A and C; aP must also be less than Pb, otherwise ef cannot be applied to the semicircle on ed.

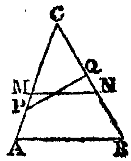
Demon. Because Cr = $\frac{1}{2}$Cl, the rhomboid Crsm = triangle Clm, and because Ca : Cr : : Cm : Cd, we have Ca . Cd = Cm . Cr, therefore rhomboid Cabd = rhomboid Crsm = triangle Clm. By reason of the parallels CB, bd, and CA, ab, the triangles aLP, dGM, bGP, are similar, and are to each other as the squares of their homologous sides aP, dM, bP: now $ed^2 = ef^2 + df^2$, by construction; and ed = Pb, ef = aP, df = dM; therefore Pb^2 = aP^2 + dM^2, or, the triangle PbG taken away from the rhomboid, is equal to the sum of the triangles aPL, dMG, added to the part CaPGd: consequently CLM = Cabd, as required. By a like process, it may be shown that aL′P, dG′M′, PbG′, are similar, and aL′P + dG′M′ = PbG′; whence PbdM′ = aL′P, and CL′M′ = Cabd, as required.

Com-

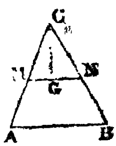
Computation. cl, cm, being known, as well as ca, aP, or ce, eP, $cr = \frac{1}{2}cl$, is known; and hence cd may be found by the proportion $ca : cr :: cm : cd$. Then $cd - ce = ed$, and $\sqrt{ed^2 - ef^2} = \sqrt{ed^2 - aP^2} = df = dM = dM'$. Thus CM is determined. Then we have $\frac{cl \,.\, cm}{CM} = CL$.

N. B. When the point is in one of the sides, as at M; then make $CL \,.\, CM \,.\, (m+n) = CA \,.\, CB \,.\, m$, or, $CL : CA :: m \,.\, CB : (m+n)CM$, and the thing is done.

5thly. By the shortest line possible.



Draw any line PQ dividing the triangle in the given ratio, and so that the summit of the triangle CPQ shall be C the *most acute* of the three angles of the triangle. Make $CM = CN$, a geometrical mean proportional between CP and CQ; so shall MN be the shortest line possible dividing the triangle in the given ratio. —The computation is evident.



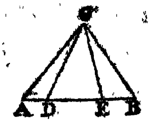
Demons. Suppose MN to be the shortest line cutting off the given triangle CMN, and $CG \perp MN$, $MN = MG + GN = CG \,.\, \cot M + CG \,.\, \cot N = CG(\cot M + \cot N)$. But, $\cot M + \cot N = \frac{\cos M}{\sin N} + \frac{\cos N}{\sin N} = \frac{\sin (M+N)}{\sin M \,.\, \sin N}$. And (equa. XVIII, ch. iii) $\sin M \,.\, \sin N = \frac{1}{2}\cos(M-N) - \frac{1}{2}\cos(M+N) = \frac{1}{2}\cos(M-N) + \frac{1}{2}\cos C$. Theref. $MN = CG \,.\, \frac{\sin (M+N)}{\frac{1}{2}\cos(M-N) + \frac{1}{2}\cos C}$; which expression is a minimum when its denominator is a maximum; that is, when $\cos(M-N)$ is the greatest possible, which is manifestly when $M - N = 0$, or $M = N$, or when the triangle CMN is isosceles. That the isosceles triangle must have the most *acute* angle for its summit, is evident from the consideration, that since $2 \triangle CMN = CG \,.\, MN$, MN varies inversely as CG; and consequently MN is shortest when CG is longest, that is, when the angle C is the most acute.

N. B. A very simple and elegant demonstration to this case is given in Simpson's Geometry: vide the book on Max. and Min. See also another demonstration at case 2d prob. 6th, below.

PROBLEM II.

To Divide a Triangle into Three Parts, having the Ratio of the quantities m, n, p.

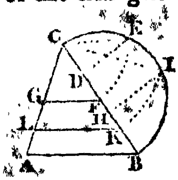
1st. By lines drawn from one angle of the triangle to the opposite side.



Divide the side AB, opposite the angle C from whence the lines are to proceed, in the given ratio at D, E; join CD, CE; and ACD, DCE, ECB, are the three triangles required. The demonstration is manifest; as is also, the computation.

If it be wished that the lines of division be the shortest the nature of the case will admit of, let them be drawn from the most obtuse angle, to the opposite or *longest* side.

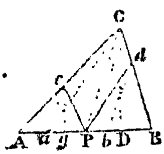
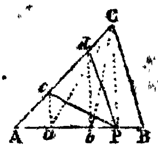
2dly. By lines parallel to one of the sides of the triangle.



Make CD : DH : HB : : $m : n : p$. Erect DE, HI, perpendicularly to CB, till they meet the semicircle described on the diameter CB, in E and I. Make CF = CE, and CK = CI. Draw GF through F, and LK through K, parallel to AB; so shall the lines GF and LK, divide the triangle ABC as required.

The demonstration and computation will be similar to those in the second case of prob. 1.

3dly. By lines drawn from a given point on one of the sides.

Fig. 1.  *Fig.* 2. 

Let P (fig. 1) be the given point, a and b the points which divide the side AB in the given ratio of m, n, p: the point P falling between a and b. Join PC, parallel to which draw ac, bd, to meet the sides AC, BC, in the points c and d: join Pc, Pd, so shall the lines cP, Pd, divide the triangle in the given ratio.

In fig. 2, where P falls nearer one of the extremities of AB than both a and b, the construction is essentially the same; the sole difference in the result is, that the points c, and d, both fall on *one* side AC of the triangle.

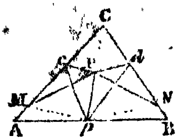
Demon. The lines Ca, Cb, divide the triangle into the given ratio, by case 1st. But by reason of the parallel lines ac, PC, bd, $\triangle$ acC = $\triangle$ acP, and $\triangle$ bdC = bdP. Therefore, in fig. 1, Aac + acP = Aac + acC, that is, AcP = AaC: and Bbd + bdP = Bbd + bdC, that is, BdP = BbC. Consequently, the remainder CcPd = Cab.—In fig. 2, AcP = AaC, and AdP = Acb; therefore cPd = acP; and ACB − AdP = ACB − ACb, that is, CBPd = CBb.

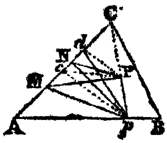
Computation. The perpendiculars cg, CD being demitted,

$\triangle$ AcP

$\triangle$ ACP : $\triangle$ ACB :: m : $m+n+p$ :: AP . cg : AB . CD. Therefore $(m+n+p)$ AP . $cg = m$. AB . CD, and $cg = \frac{m \cdot \text{AB} \cdot \text{CD}}{(m+n+p)\ \text{AP}}$. The line cg being thus known, we soon find Ac; for CD : AC :: cg : Ac $= \frac{\text{AC} \cdot cg}{\text{CD}} = \frac{m \cdot \text{AB} \cdot \text{AC}}{(m+n+p)\ \text{AP}}$. Indeed this expression may be deduced more simply; for, since ACB : AcP :: AC . AB : Ac.AP :: $m+n+p$: m, we have $(m+n+p)$ Ac.AP $= m$.AB.AC, and Ac $= \frac{m \cdot \text{AB} \cdot \text{AC}}{(m+n+p)\ \text{AP}}$. By a like process is obtained, in fig. 1, Bd $= \frac{p \cdot \text{AB} \cdot \text{BC}}{(m+n+p)\ \text{PB}}$; and, in fig. 2, A$d$ $= \frac{(m+n)\ \text{AB} \cdot \text{AC}}{(m+n+p)\ \text{AP}}$.

4thly. By lines drawn from a given point P *within* the triangle.





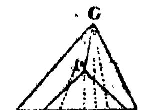
Const. Through P and C draw the line CPp, and let the triangle be divided into the given ratio by lines pc, pd, drawn from p to intersect AC, BC, or either of them; according to the method described in case 3 of this problem. Through P draw Pc, Pd, and respectively parallel to them, from p draw the lines pM, pN: join PM, PN; so shall these lines with Pp, divide the triangle in the given ratio.

Demon. The triangles cPM, cPp, are manifestly equal, as are also dPN, dPp; therefore CPM = Cpc, and CPN = Cpd; whence also, in fig. 1, CNPM = Cdpc, and, in fig. 2, CBpPN = CBpd.

Comput. Since CP . CN = Cp . Cd, we have CN $= \frac{\text{C}p \cdot \text{C}d}{\text{CP}}$.

In like manner CM $= \frac{\text{C}p \cdot \text{C}c}{\text{CP}}$.

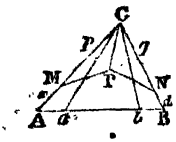
Remark. It will generally be best to contrive that the *smallest* share of the triangle shall be laid off nearest the vertex C of the triangle, in order to ensure the possibility of the construction. Even this precaution however may sometimes fail, of ensuring the construction by the method above given: when this happens, proceed thus:

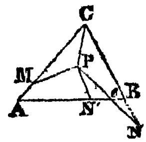


By case 1, draw the lines Cd, Ce, from the vertex C to the opposite side AB, to divide the triangle in the given ratio. Upon AB set off any where MN, so that MN : AB :: Pp (the perp. from P on AB) : Cp, the altitude of the triangle. If MP and PN are together

gether to be the least possible, then set off $\frac{1}{2}$ MN on each side the point p: so will the triangle MPN be isosceles, and its perimeter (with the given base and area) a minimum.

5thly. By lines, *one* of which is drawn *from* a given angle *to* a given point, which is also the point of concourse of the other two lines.





Const. By case 1st draw the lines Ca, Cb, dividing the triangle in the given ratio, and so that the smaller portions shall lie nearest the angles A and B (unless the conditions of the division require it to be otherwise). From P and a demit upon AC the perpendiculars Pp, ac; and from P and b, on BC, the perpendiculars Pq, bd. Make CM : CA : : ac : Pp, and CN : CB : : bd : Pq. Draw PM, PN, which, with CP, will divide the triangle as required.

When the perpendicular from b or from a, upon BC or AC, is longer than the corresponding perpendicular from P, the point N or M will fall further from C than B or A does. Suppose it to be N': then make N'e : eB : : Ne : eP, and draw PN' for the line of division.

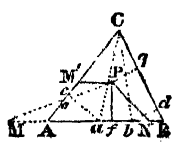
The demonstration of all this is too obvious to need tracing here.

Comput. The perp. $ca = Aa \cdot \sin A$; and $CM = \frac{CA \cdot ac}{Pp}$.

$$bd = Bb \cdot \sin B; \text{ and } CN = \frac{CB \cdot bd}{Pq}.$$

6thly. By lines, one of which falls from the given point of concourse of all three, upon a given side, in a given angle.

Suppose the given angle to be a right angle, and Pf the given perpendicular: which will simplify the operation, though the principles of construction will be the same.



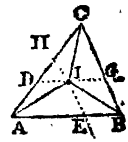
Const. Let Ca, Cb, divide the triangle in the given ratio. Make fN : CB : : bd : Pf, and fM : CA : : ac : Pf; and draw PN, PM, thus forming two triangles PfN, PfM, equal to CbB, CaA respectively. If N fall between f and B, and M between A and f, this construction manifestly effects the division. But if one of the points, suppose M, falls beyond the corresponding point A, the line PM intersecting

intersecting AC in e: then make $\text{M}'e : e\text{A} :: e\text{M} : e\text{P}$, and draw PM': so shall $\text{P}f$, PM', PN, divide the triangle as required.

Comput. Here ca and bd are found as in case 5th; and hence $f\text{N} = \frac{\text{CB} \cdot bd}{\text{P}f}$; and $f\text{M} = \frac{\text{CA} \cdot ac}{\text{P}f}$. Then $\text{PM} = \sqrt{(\text{M}f^2 + f\text{P}^2)}$, and $\frac{\text{P}f}{\text{PM}} = \sin \text{M}$. Also $180° - (\text{M} + \text{A}) = \text{M}e\text{A}$. Then $\sin \text{M}e\text{A} : \sin \text{M} : \sin \text{A} \propto \text{MA} (= \text{M}f - \text{A}f) : \text{A}e : \text{M}e$. Again $\text{P}e = \text{PM} - \text{M}e$; and lastly $\text{M}'e = \frac{\text{A}e \cdot e\text{M}}{e\text{P}}$.

Here also the demonstration is manifest.

7thly. By lines drawn from the angles to meet in a determinate point.



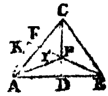
Construc. On one of the sides, as AC, set off AD, so that $\text{AD} : \text{AC} :: m : m + n + p$. And on any other, as AB, set off BE, so that $\text{BE} : \text{BC} :: n : m + n + p$. Through D draw DG parallel to AB; and through E, EH parallel to BC; to their point of intersection I draw the lines AI, BI, CI, which will divide the triangle ABC into the portions required.

Demon. Any triangle whose base is AB, and whose vertex falls in DG parallel to it, will manifestly be to ABC, as AD to AC, or as m to $m + n + p$: so also, any triangle whose base is BC, and whose vertex falls in EH parallel to it, will be to ABC, as BE to BA, that is, as n to $m + n + p$.

Thus we have $\text{AIB} : \text{ACB} :: m : m + n + p$,
and . . . $\text{BIC} : \text{ACB} :: n : m + n + p$,
therefore . $\text{AIB} : \text{BIC} :: m : n$.

And the first two proportions give, by composition, $\text{AIB} + \text{BIC} : \text{ACB} :: m + n : m + n + p$; and by division, $\text{ACB} - (\text{AIB} + \text{BIC}) : \text{ACB} :: m + n + p - (m + n) : m + n + p$, or $\text{AIC} : \text{ACB} :: p : m + n + p$, consequently $\text{AIB} : \text{BIC} : \text{AIC} \propto m : n : p$.

Comput. $\text{BE} = \text{GI} = \frac{n \cdot \text{AB}}{m + n + p}$; $\text{BG} = \frac{m \cdot \text{BC}}{m + n + p}$; angle BGI = 2 right angles − B. Hence, in the triangle BGI, there are known two sides and the included angle, to find the third side BI.



Remark. When $m = n = p$, the construction becomes simpler. Thus: from the vertex draw CD to bisect AB; and from B draw BE in like manner to the middle of AC: the point of intersection I of the lines CD, BE, will be the point sought.

For, on BE and BE produced, demit, from the angles C and A, the perpendiculars CI, AK: then the triangles CEI, AEK, are equal in all respects, because $\text{AE} = \text{CE}$, $\text{KAE} = \text{ICE}$, and the

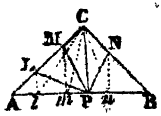
the angles at E are equal. Hence AK = CI. But these are the perpendicular altitudes of the triangles BPC, BPA, which have the common base BP. Consequently those two triangles are equal in area. In a similar manner it may be proved, that APC = APB or CPB: therefore these three triangles are equal to each other, and the lines PA, PB, PC, trisect the △ ABC.

PROBLEM III.

To Divide a Triangle into Four Parts, having the Proportion of the Quantities *m*, *n*, *p*, *q*.

This, like the former problems, might be divided into several cases, the consideration of all which would draw us to a very great length, and which is in great measure unnecessary, because the method will in general be suggested immediately on contemplating the method of proceeding in the analogous case of the preceding problem. We shall therefore only take one case, namely, that in which the lines of division must all be drawn from a given point in one of the sides.

Let P be the given point in the side AB.



Let the points *l*, *m*, *n*, divide the base AB in the given proportion; so will the lines C*l*, C*m*, C*n*, divide the surface of the triangle in the same proportion. Join CP, and parallel to it draw, from *l*, *m*, *n*, the lines *l*L, *m*M, *n*N, to cut the other two sides of the triangle in L, M, N. Draw PL, PM, PN, which will divide the triangle as required.

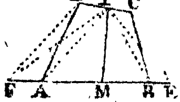
The demonstration is too obvious to need tracing throughout: for the triangles L*l*P, L*l*C, having the same base L*l*, and lying between the same two parallels L*l*, CP, are equal; to each of these adding the triangle AL*l*, there results ALP = AC*l*. And in like manner the truth of the whole construction may be shown.

The computation may be conducted after the manner of that in case 3d prob. 2.

PROBLEM IV.

To Divide a Quadrilateral into Two Parts having a Given Ratio, *m* : *n*.

1st. By a line drawn from any point in the perimeter of the figure.



Construc. From P draw lines PA, PB, to the opposite angles A, B. Through D draw DF parallel to PA, to meet BA produced in F: and through C draw CE parallel to PB to meet AB produced in E.

Divide

Divide FE in M, in the given ratio of m to n: join P, M; so shall the line PM divide the quadrilateral as required.

Demon. That the triangle FPE is equal to the quadrangle ABCD, may be shown by the same process as is used to demonstrate the construction of prob. 36, Geometry; of which, in fact, this is only a modification. And the line PM evidently divides FPE in the given ratio. But FPM = ADPM, and EPM = BCPM: therefore PM divides the quadrangle also in the given ratio.

Remark 1. If the line PM cut either of the sides AD, BC, then its position must be changed by a process similar to that described in the 5th and 6th cases of the last problem.

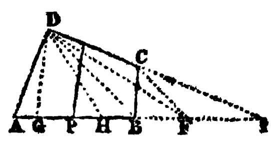
Remark 2. The quadrilateral may be divided into three, four, or more parts, by a similar method, being subject however to the restriction mentioned in the preceding remark.

Remark 3. The same method may obviously be used when the given point P is in one of the angles of the figure.

Comput. Suppose I to be the point of intersection of the sides DC and AB, produced; and let the part of the quadrilateral laid off towards I, be to the other, as n to m. Then we have $\text{IM} = \frac{n(\text{ID} \cdot \text{IA} - \text{IB} \cdot \text{IC})}{(m+n)\,\text{IP}}$. As to the distances DI, AI, (since the angles at A and D, and consequently that at I, are known), they are easily found from the proportionality of the sides of triangles to the sines of their opposite angles.

2dly. By a line drawn parallel to a given line.

Construc. Produce DC, AB, till they meet, as at I. Join DB, parallel to which draw CF. Divide AF in the given ratio in H. Through D draw DG parallel to the given line. Make IP a mean proportional between IH, IG; through P draw PM parallel to GD: so shall PM divide the quadrilateral ABCD as required.



Demon. It is evident, from the transformation of figures, so often resorted to in these problems, that the triangle ADF = quadrilateral ABCD (th. 36 Geom.): and that DH divides the triangle ADF in the given ratio, is evident from prob 1 case 1. We have only then to demonstrate that the triangle IHD is equal to the triangle IPM, for in that case HDF will manifestly be equal to BCMP. Now, by construction, IH : IP :: IP : IG :: (by the parallels) IM : ID; whence, by making the products of the means and extremes equal, we have ID . IH = IP . IM; but when the products of the sides about the

the equal angles of two triangles having a common angle are equal, those triangles are equal; therefore △ IHD = △ IPM. Q. E. D.

Comput. In the triangles ADI, ADG, are given all the angles, and the side AD; whence AI, AG, DI, and IC, = DI − DC, become known. In the triangle IFC, all the angles and the side IC are known; whence IF becomes known, as well as FH, since AH : HF : : m : n. Lastly, IP = $\sqrt{}$(IH . IG), and IG : ID : : IP : IM.

Cor. 1. When the line of division PM is to be perpendicular to a side, or parallel to a given side; we have only to draw DG accordingly: so that those two cases are included in this.

Cor. 2. When the line PM is to be the shortest possible, it must cut off an isosceles triangle towards the acutest angle; and in that case IG must evidently be equal to ID.

3dly. By a line drawn through a given point.

The method will be the same as that to case 4th prob. 1, and therefore need not be repeated here.

Scholium. If a quadrilateral were to be divided into four parts in a given proportion, m, n, p, q: we must first divide it into two parts having the ratio of $m + n$, to $p + q$; and then each of the quadrangles so formed into their respective ratios, of m to n, and p to q.

PROBLEM V.

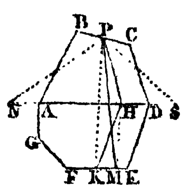
To Divide a Pentagon into Two Parts having a Given Ratio, from a Given Point in one of the Sides.

Reduce the pentagon to a triangle by prob. 37, Geometry, and divide this triangle in the given ratio by case 1 prob. 1.

PROBLEM VI.

To Divide *any* Polygon into Two Parts having a Given Ratio.

1st. From a given point in the perimeter of the polygon.



Construc. Join any two opposite angles A, D, of the polygon, by the line AD. Reduce the part ABCD into an equivalent triangle NPS, whose vertex shall be the given point P, and base AD produced: an operation which may be performed at once, if the portion ABCD be quadrangular; or by several operations (as from 8 sides to 6, from 6 to 4, &c,) if the sides be more than four. Divide the triangle NPS into two parts having the given ratio, by the line PH. In like manner, reduce

ADEFGA

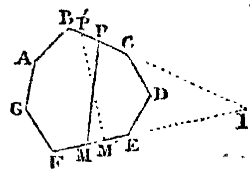
ADEFGA into an equivalent triangle having H for its vertex, and FE produced for its base; and divide this triangle into the given ratio by a line from H, as HK. The compound line PHK will manifestly divide the whole polygon into two parts having the given ratio. To reduce this to a right line, join PK, and through H draw HM parallel to it; join PM; so will the right line PM divide the polygon as required, provided M fall between F and E. If it do not, the reduction may be completed by the process described in cases 5th and 6th prob. 2d.

All this is too evident to need demonstration.

Remark. There is a *direct* method of solving this problem, without subdividing the figure; but as it requires the computation of the area, it is not given here.

2dly. By the shortest line possible.

Construc. From any point P′, in one of those two sides of the polygon which, when produced, meet in the most acute angle I, draw a line P′M′, to the other of those sides (EF), dividing the polygon in the given ratio. Find the points P and M, so that IP or IM shall be a mean proportional between IP′, IM′; then will PM be the line of division required.

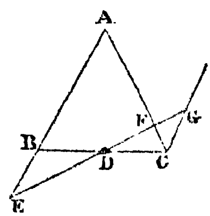


The demonstration of this is the same as has been already given, at case 5 prob. 1. Those, however, who wish for a proof, independent of the arithmetic of sines, will not be displeased to have the additional demonstration below.

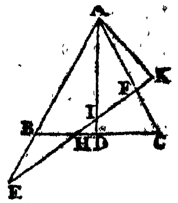
The *shortest* line which, with two other lines given in position, includes a given area, will make equal angles with those two lines, or with the segments of them it cuts off from an isosceles triangle.

Let the two triangles ABC, AEF, having the common angle A, be equal in surface, and let the former triangle be isosceles, or have AB = AC; then is BC shorter than EF.

First, the oblique base EF cannot pass through D, the middle point of BC, as in the annexed figure. For, drawing CG parallel to AB, to meet EF produced in G. Then the two triangles DBE, DCG are identical, or mutually equal in all respects. Consequently the triangle DCF is less than DBE, and therefore ABC less than AEF.



EF must therefore cut BC in some point H between B and D, and cutting the perp. AD in some point I above D, as in the 2d fig.



2d fig. Upon EF (produced if necessary) demit the perp. AK. Then, in the right angled Δ AIK, the perp. AK is less than the hypothenuse AI, and therefore much more less than the other perp. AD. But, of equal triangles, that which has the greatest perpendicular, has the least base. Therefore the base BC is less than the base EF. Q. E. D.

This series of problems might have been extended much further; but the preceding will furnish a sufficient variety, to suggest to the student the best method to be adopted in almost any other case that may occur. The following practical examples are subjoined by way of exercise.

Ex. 1. A triangular field, whose sides are 20, 18, and 16 chains, is to have a piece of 4 acres in content fenced off from it, by a right line drawn from the most obtuse angle to the opposite side. Required the length of the dividing line, and its distance from either extremity of the line on which it falls?

Ex. 2. The three sides of a triangle are 5, 12, and 13. If two-thirds of this triangle be cut off by a line drawn parallel to the longest side, it is required to find the length of the dividing line, and the distance of its two extremities from the extremities of the longest side.

Ex. 3. It is required to find the length and position of the shortest possible line, which shall divide, into two equal parts, a triangle whose sides are 25, 24, and 7 respectively.

Ex. 4. The sides of a triangle are 6, 8, and 10: it is required to cut off nine-sixteenths of it, by a line that shall pass through the centre of its inscribed circle.

Ex. 5. Two sides of a triangle, which include an angle of 70°, are 14 and 17 respectively. It is required to divide it into three equal parts, by lines drawn parallel to its longest side.

Ex. 6. The base of a triangle is 112·65, the vertical angle 57°57′, and the difference of the sides about that angle is 8. It is to be divided into three equal parts, by lines drawn from the angles to meet in a point within the triangle. The lengths of those lines are required.

Ex. 7. The legs of a right-angled triangle are 28 and 45. Required the lengths of lines drawn from the middle of the hypothenuse, to divide it into four equal parts.

Ex.

Ex. 8. The length and breadth of a rectangle are 15 and 9. It is proposed to cut off one-fifth of it, by a line which shall be drawn from a point on the longest side at the distance of 4 from a corner.

Ex. 9. A regular hexagon, each of whose sides is 12, is to be divided into four equal parts, by two equal lines; both passing through the centre of the figure. What is the length of those lines when a minimum?

Ex. 10. The three sides of a triangle are 5, 6, and 7. How may it be divided into four equal parts, by two lines which shall cut each other perpendicularly?

*** The student will find that some of these examples will admit of two answers.

CHAPTER VIII.

ON THE NATURE AND SOLUTION OF EQUATIONS IN GENERAL.

1. In order to investigate the general properties of the higher equations, let there be assumed between an unknown quantity x, and given quantities a, b, c, d, an equation constituted of the continued product of uniform factors: thus

$$(x-a)\times(x-b)\times(x-c)\times(x-d)=0.$$

This, by performing the multiplications, and arranging the final product according to the powers or dimensions of x, becomes

$$x^4 \left.\begin{array}{l} -a \\ -b \\ -c \\ -d \end{array}\right\} x^3 \left.\begin{array}{l} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{array}\right\} x^2 \left.\begin{array}{l} -abc \\ -abd \\ -acd \\ -bcd \end{array}\right\} x + abcd = 0. \quad . \quad . \quad . \quad (A)$$

Now it is obvious that the assemblage of terms which compose the first side of this equation may become equal to nothing in four different ways; namely, by supposing either $x=a$, or $x=b$, or $x=c$, or $x=d$; for in either case one or other of the factors $x-a$, $x-b$, $x-c$, $x-d$, will be equal to nothing, and nothing multiplied by any quantity whatever will give *nothing* for the product. If any other value e be put for x, then none of the factors $e-a$, $e-b$, $e-c$, $e-d$, being equal to nothing, their continued product cannot be equal to nothing. There are therefore, in the proposed equation, four roots

roots or values of x; and that which characterizes these roots is, that on substituting each of them successively instead of x, the aggregate of the terms of the equation vanishes by the opposition of the signs $+$ and $-$.

The preceding equation is only of the fourth power or degree; but it is manifest that the above remark applies to equations of higher or lower dimensions: viz, that in general an equation of any degree whatever has as many roots as there are units in the exponent of the highest power of the unknown quantity, and that each root has the property of rendering, by its substitution in place of the unknown quantity, the aggregate of all the terms of the equation equal to nothnig.

It must be observed that we cannot have all at once $x=a$, $x=b$, $x=c$, &c, for the roots of the equation; but that the particular equations $x-a=0$, $x-b=0$, $x-c=0$, &c, obtain only in a *disjunctive* sense. They exist as factors in the same equation, because algebra gives, by one and the same formula, not only the solution of the particular problem from which that formula may have originated, but also the solution of all problems which have similar conditions. The different roots of the equation satisfy the respective conditions: and those roots may differ from one another, by their *quantity*, and by their *mode* of existence.

It is true, we say frequently that the roots of an equation are $x=a$, $x=b$, $x=c$, &c, as though those values of x existed conjunctively; but this manner of speaking is an abbreviation, which it is necessary to understand in the sense explained above.

2. In the equation A, all the roots are positive; but if the factors which constitute the equation had been $x+a$, $x+b$, $x+c$, $x+d$, the roots would have been negative or subtractive. Thus

$$x^4 \left.\begin{matrix} +a \\ +b \\ +c \\ +d \end{matrix}\right\} x^3 \left.\begin{matrix} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{matrix}\right\} x^2 \left.\begin{matrix} +abc \\ +abd \\ +acd \\ +bcd \end{matrix}\right\} x + abcd = 0 \ldots \text{(B)}$$

has negative roots, those roots being $x=-a$, $x=-b$, $x=-c$, $x=-d$: and here again we are to apply them disjunctively.

3. Some equations have their roots in part positive, in part negative. Such is the following:

$$x^3 \left.\begin{matrix} -a \\ -b \\ +c \end{matrix}\right\} x^2 \left.\begin{matrix} +ab \\ -ac \\ -bc \end{matrix}\right\} x + abc = 0 \ldots\ldots\ldots \text{(C)}$$

Here

Here are the two positive roots viz, $x = a$, $x = b$; and one negative root, viz, $x = -c$: the equation being constituted of the continued product of the three factors, $x - a = 0$, $x - b = 0$, $x + c = 0$.

From an inspection of the equations A, B, C, it may be inferred, that a complete equation consists of a number of terms exceeding by *unity* the number of its roots.

4. The preceding equations have been considered as formed from equations of the first degree, and then each of them contains so many of these constituent equations as there are units in the exponent of its degree. But an equation which exceeds the second dimension, may be considered as composed of one or more equations of the second degree, or of the third, &c, combined, if it be necessary, with equations of the first degree, in such manner, that the product of all those constituent equations shall form the proposed equation. Indeed, when an equation is formed by the successive multiplication of several simple equations, quadratic equations, cubic equations, &c, are formed; which of course may be regarded as factors of the resulting equation.

5. It sometimes happens that an equation contains imaginary roots; and then they will be found also in its constituent equations. This class of roots always enters an equation by pairs; because they may be considered as containing, in their expression at least, one *even* radical placed before a negative quantity; and because an *even* radical is necessarily preceded by the double sign $\pm$. Let, for example, the equation be $x^4 - (2a - 2c)x^3 + (a^2 + b^2 - 4ac + c^2 + d^2)x^2 + (2a^2c + 2b^2c - 2ac^2 - 2ad^2)x + (a^2 + b^2) . (c^2 + d^2) = 0$. This may be regarded as constituted of the two subjoined quadratic equations, $x^2 - 2ax + a^2 + b^2 = 0$, $x^2 + 2cx + c^2 + d^2 = 0$: and each of these quadratics contains two imaginary roots; the first giving $x = a \pm b\sqrt{-1}$, and the second $x = -c \pm d\sqrt{-1}$.

In the equation resulting from the product of these two quadratics, the coefficients of the powers of the unknown quantity, and of the last term of the equation, are real quantities, though the constituent equations contain imaginary quantities; the reason is, that these latter disappear by means of addition and multiplication.

The same will take place in the equation $(x - a) . (x + b) . (x^2 + 2cx + c^2 + d^2) = 0$, which is formed of two equations of the first degree, and one equation of the second whose roots are imaginary.

These remarks being premised, the subsequent general theorems will be easily established.

THEOREM

THEOREM I.

Whatever be the Species of the Roots of an Equation, when the Equation is arranged according to the Powers of the Unknown Quantity, if the First Term be positive, and have unity for its Coefficient, the following Properties may be traced:

I. The first term of the equation is the unknown quantity raised to the power denoted by the number of roots.

II. The second term contains the unknown quantity raised to a power less than the former by unity, with a coefficient equal to the sum of the roots taken with contrary signs.

III. The third term contains the unknown quantity raised to a power less by 2 than that of the first term, with a coefficient equal to the sum of all the products which can be formed by multiplying all the roots two and two.

IV. The fourth term contains the unknown quantity raised to a power less by 3 than that of the first term, with a coefficient equal to the sum of all the products which can be made by multiplying any three of the roots with contrary signs.

V. And so on to the last term, which is the continued product of all the roots taken with contrary signs.

All this is evident from inspection of the equations exhibited in arts. 1, 2, 3, 5.

Cor. 1. Therefore an equation having all its roots real, but some positive the others negative, will want its second term when the sum of the positive roots is equal to the sum of the negative roots. Thus, for example, the equation c will want its second term, if $a + b = c$.

Cor. 2. An equation whose roots are all imaginary, will want the second term, if the sum of the real quantities which enter into the expression of the roots, is partly positive, partly negative, and has the result reduced to nothing, the imaginary parts mutually destroying each other by addition in each pair of roots. Thus, the first equation of art. 5 will want the second term if $-2a + 2c = 0$, or $a = c$. The second equation of the same article, which has its roots partly real, partly imaginary, will want the second term if $b - a + 2c = 0$, or $a - b = 2c$.

Cor. 3. An equation will want its third term, if the sum of the products of the roots taken two and two, is partly positive, partly negative, and these mutually destroy each other.

Remark. An *incomplete* equation may be thrown into the form of *complete* equations, by introducing, with the coefficient *a cypher*, the absent powers of the unknown quantity: thus, for

for the equation $x^3 + r = 0$, may be written $x^3 + 0x^2 + 0x + r = 0$. This in some cases will be useful.

Cor. 4. An equation with positive roots may be transformed into another which shall have negative roots of the same value, and reciprocally. In order to this, it is only necessary to change the signs of the alternate terms, beginning with the second. Thus, for example, if instead of the equation $x^3 - 8x^2 + 17x - 10 = 0$, which has three positive roots 1, 2, and 5, we write $x^3 + 8x^2 + 17x + 10 = 0$, this latter equation will have three negative roots $x = -1$, $x = -2$, $x = -5$. In like manner, if instead of the equation $x^3 + 2x^2 - 13x + 10 = 0$, which has two positive roots $x = 1$, $x = 2$, and one negative root $x = -5$, there be taken $x^3 - 2x^2 - 13x - 10 = 0$, this latter equation will have two negative roots, $x = -1$, $x = -2$, and one positive root $x = 5$.

In general, if there be taken the two equations, $(x-a) \times (x-b) \times (x-c) \times (x-d) \times \&c = 0$, and $(x+a) \times (x+b) \times (x+c) \times (x+d) \times \&c = 0$, of which the roots are the same in magnitude, but with different signs: if these equations be developed by actual multiplication, and the terms arranged according to the powers of x, as in arts. 1, 2; it will be seen that the second terms of the two equations will be affected with different signs, the third terms with like signs, the fourth terms with different signs, &c.

When an equation has not all its terms, the deficient terms must be supplied by cyphers, before the preceding rule can be applied.

Cor. 5. The sum of the roots of an equation, the sum of their squares, the sum of their cubes, &c, may be found without knowing the roots themselves. For, let an equation of any degree or dimension, m, be $x^m + fx^{m-1} + gx^{m-2} + hx^{m-3} + \&c = 0$, its roots being a, b, c, d, &c. Then we shall have,

1st. The sum of the first powers of the roots, that is, of the roots themselves, or $a + b + c + \&c = -f$; since the coefficient of the unknown quantity in the second term, is equal to the sum of the roots taken with different signs.

2dly. The sum of the squares of the roots, is equal to the square of the coefficient of the second term made less by twice the coefficient of the third term: viz, $a^2 + b^2 + c^2 + \&c = f^2 - 2g$. For, if the polynomial $a + b + c + \&c$, be squared, it will be found that the square contains the sum of the squares of the terms a, b, c, &c, *plus* twice the sum of the products formed by multiplying two and two all the roots a, b, c, &c. That is, $(a + b + c + \&c)^2 = a^2 + b^2 + c^2 + \&c + 2(ab + ac + bc + \&c)$. But it is obvious, from equa. A, B, that

that $(a+b+c+\&c)^2, =f^2$ and $(ab+ac+bc+\&c)=g$. Thus we have $f^2=(a^2+b^2+c^2+\&c)+2g$; and consequently $a^2+b^2+c^2+\&c=f^2-2g$.

3dly. The sum of the cubes of the roots, is equal to 3 times the rectangle of the coefficient of the second and third terms, made less by the cube of the coefficient of the second term, and 3 times the coefficient of the fourth term: viz, $a^3+b^3+c^3+\&c=-f^3+3fg-3h$. For we shall by actual involution, have $(a+b+c+\&c)^3=a^3+b^3+c^3+\&c+3(a+b+c)\times(ab+ac+bc)-3abc$. But $(a+b+c+\&c)^3=-f^3$, $(a+b+c+\&c)\times(ab+ac+bc+\&c)=-fg$, $abc=-h$. Hence therefore, $-f^3=a^3+b^3+c^3+\&c-3fg+3h$; and consequently, $a^3+b^3+c^3+\&c=-f^3+3fg-3h$. And so on, for other powers of the roots.

THEOREM II.

In Every Equation, which contains only Real Roots:

I. If all the roots are positive, the terms of the equation will be $+$ and $-$ alternately.

II. If all the roots are negative, all the terms will have the sign $+$.

III. If the roots are partly positive, partly negative, there will be as many positive roots as there are *variations* of signs, and as many negative roots as there are *permanencies* of signs; these variations and permanencies being observed from one term to the following through the whole extent of the equation.

In all these, either the equations are complete in their terms, or they are made so.

The first part of this theorem is evident from the examination of equation A; and the second from equation B.

To demonstrate the third, we revert to the equation C (art. 3), which has two positive roots, and one negative. It may happen that either $c > a+b$, or $c < a+b$.

In the first case, the second term is positive, and the third is negative; because, having $c > a+b$, we shall have $ac+bc > (a+b)^2 > ab$. And, as the last term is positive, we see that from the first to the second there is a permanence of signs; from the second to the third a variation of signs; and from the third to the fourth another variation of signs. Thus there are two variations and one permanence of signs; that is, as many variations as there are positive roots, and as many permanencies as there are negative roots.

In the second case, the second term of the equation is negative, and the third may be either positive or negative. If that

that term is positive, there will be from the first to the second a variation of signs; from the second to the third another variation; from the third to the fourth a permanence; making in all two variations and one permanence of signs. If the third term be negative; there will be one variation of signs from the first to the second; one permanence from the second to the third; and one variation from the third to the fourth; thus making again two variations and one permanence. The number of variations of signs therefore, in this case as well as in the former, is the same as that of the positive roots; and the number of permanencies, the same as that of the negative roots.

Corol. Whence it follows, that if it be known, by any means whatever, that an equation contains only real roots, it is also known how many of them are positive, and how many negative. Suppose, for example, it be known that, in the equation $x^5 + 3x^4 - 23x^3 - 27x^2 + 166x - 120 = 0$, all the roots are real: it may immediately be concluded that there are *three* positive and *two* negative roots. In fact this equation has the three positive roots $x = 1$, $x = 2$, $x = 3$; and two negative roots, $x = -4$, $x = -5$.

If the equation were incomplete, the absent terms must be supplied by adopting cyphers for coefficients, and those terms must be marked with the ambiguous sign $\pm$. Thus, if the equation were

$$x^5 - 20x^3 + 30x^2 + 19x - 30 = 0,$$

all the roots being real, and the second term wanting, it must be written thus:

$$x^5 \pm 0x^4 - 20x^3 + 30x^2 + 19x - 30 = 0.$$

Then it will be seen that, whether the second term be positive or negative, there will be 3 variations and 2 permanencies of signs: and consequently the equation has 3 positive and 2 negative roots. The roots in fact are, 1, 2, 3, -1, -5.

This rule only obtains with regard to equations whose roots are real. If, for example, it were inferred that, because the equation $x^2 + 2x + 5 = 0$ had two permanencies of signs, it had two negative roots, the conclusion would be erroneous: for both the roots of this equation are imaginary.

THEOREM III.

Every Equation may be Transformed into Another whose Roots shall be Greater or Less by a Given Quantity.

In any equation whatever, of which x is unknown, (the equations A, B, C, for example) make $x = z + m$, z being a new unknown quantity, m any given quantity, positive or

 negative:

negative: then substituting, instead of x and its powers, their values resulting from the hypothesis that $x = z + m$; so shall there arise an equation, whose roots shall be greater or less than the roots of the primitive equation, by the assumed quantity m.

Corol. The principal use of this transformation is, to take away any term out of an equation. Thus, to transform an equation into one which shall want the *second* term, let m be so assumed that $nm - a = 0$, or $m = \frac{a}{n}$, n being the index of the highest power of the unknown quantity, and a the coefficient of the second term of the equation, with its sign changed: then, if the roots of the transformed equation can be found, the roots of the original equation may also be found, because $x = z + \frac{a}{n}$.

THEOREM IV.

Every Equation may be Transformed into Another, whose Roots shall be Equal to the Roots of the First Multiplied or Divided by a Given Quantity.

1. Let the equation be $z^3 + az^2 + bz + c = 0$: if we put $fz = x$, or $z = \frac{x}{f}$, the transformed equation will be $x^3 + fax^2 + f^2bx + f^3c = 0$, of which the roots are the respective products of the roots of the primitive equation multiplied into the quantity f.

By means of this transformation, an equation with fractional quantities, may be changed into another which shall be free from them. Suppose the equation were $z^3 + \frac{az^2}{g} + \frac{bz}{h} + \frac{d}{k} = 0$: multiplying the whole by the product of the denominators, there would arise $ghkz^3 + hkaz^2 + gkbz + ghd = 0$: then assuming $ghkz = x$, or $z = \frac{x}{ghk}$, the transformed equa. would be $x^3 + hkax^2 + g^2k^2hbx + g^3k^3h^2d = 0$.

The same transformation may be adopted, to exterminate the radical quantities which affect certain terms of an equation. Thus, let there be given the equation $z^3 + az^2\sqrt{k} + bz + c\sqrt{k}$: make $z\sqrt{k} = x$; then will the transformed equation be $x^3 + akx^2 + bkx + ck^2 = 0$, in which there are no radical quantities.

2. Take, for one more example, the equation $z^3 + az^2 + bz + c = 0$. Make $\frac{z}{f} = x$; then will the equation be transformed to $x^3 + \frac{ax^2}{f} + \frac{bx}{f^2} + \frac{c}{f^3} = 0$, in which the roots are

are equal to the quotients of those of the primitive equations divided by f.

It is obvious that, by analogous methods, an equation may be transformed into another, the roots of which shall be to those of the proposed equation, in any required ratio. But the subject need not be enlarged on here. The preceding succinct view will suffice for the usual purposes, so far as relates to the nature and chief properties of equations. We shall therefore conclude this chapter with a summary of the most useful rules for the solution of equations of different degrees, besides those already given in the first volume.

I. *Rules for the Solution of Quadratics by Tables of Sines and Tangents.*

1. If the equation be of the form $x^2 + px = q$:

Make $\tan A = \frac{2}{p}\sqrt{q}$; then will the two roots be,

$x = + \tan \frac{1}{2}A \sqrt{q} \ldots\ldots x = - \cot \frac{1}{2} A \sqrt{q}$.

2. For quadratics of the form $x^2 - px = q$.

Make, as before, $\tan A = \frac{2}{p}\sqrt{q}$: then will

$x = - \tan \frac{1}{2}A \sqrt{q} \ldots\ldots x = + \cot \frac{1}{2}A \sqrt{q}$.

3. For quadratics of the form $x^2 + px = - q$.

Make $\sin A = \frac{2}{p}\sqrt{q}$: then will

$x = - \tan \frac{1}{2}A \sqrt{q} \ldots\ldots x = - \cot \frac{1}{2}A \sqrt{q}$.

4. For quadratics of the form $x^2 - px = - q$.

Make $\sin A = \frac{2}{p}\sqrt{q}$: then will

$x = + \tan \frac{1}{2}A \sqrt{q} \ldots\ldots x = + \cot \frac{1}{2}A \sqrt{q}$.

In the last two cases, if $\frac{2}{p}\sqrt{q}$ exceed unity, $\sin A$ is imaginary, and consequently the values of x.

The logarithmic application of these formulæ is very simple. Thus, in case 1st. Find A by making

$$10 + \log 2 + \tfrac{1}{2} \log q - \log p = \log \tan A.$$

$$\text{Then } \log x = \begin{cases} + \log \tan \frac{1}{2}A + \frac{1}{2} \log q - 10. \\ - (\log \cot \frac{1}{2}A + \frac{1}{2} \log q - 10). \end{cases}$$

Note. This method of solving quadratics, is chiefly of use when the quantities p and q are large integers, or complex fractions.

II. *Rules for the Solution of Cubic Equations by Tables of Sines, Tangents, and Secants.*

1. For cubics of the form $x^3 + px \pm q = 0$.

Make $\tan B = \frac{\frac{1}{3}p}{q} \cdot 2\sqrt{\frac{1}{3}p} \ldots\ldots \tan A = \sqrt[3]{\tan \frac{1}{2}B}$.

Then $x = \mp \cot 2A \cdot 2\sqrt{\frac{1}{3}p}$.

2. For

2. For cubics of the form $x^3 - px \pm q = 0$.

Make $\sin B = \frac{\frac{1}{2}p}{q} \cdot 2\sqrt{\frac{1}{3}p}$ . . . . $\tan A = \sqrt[3]{\tan \frac{1}{2}B}$.

$$\text{Then } x = \mp \operatorname{cosec} 2A \cdot 2\sqrt{\tfrac{1}{3}p}.$$

Here, if the value of $\sin B$ should exceed unity, B would be imaginary, and the equation would fall in what is called the *irreducible case* of cubics. In that case we must make $\operatorname{cosec} 3A = \frac{\frac{1}{2}p}{q} \cdot 2\sqrt{\frac{1}{3}p}$; and then the three roots would be

$$x = \pm \sin A \cdot 2\sqrt{\tfrac{1}{3}p}.$$
$$x = \pm \sin (60^\circ - A) \cdot 2\sqrt{\tfrac{1}{3}p}.$$
$$x = \pm \sin (60^\circ + A) \cdot 2\sqrt{\tfrac{1}{3}p}.$$

If the value of $\sin B$ were 1, we should have $B = 90^\circ$, $\tan A = 1$; therefore $A = 45^\circ$, and $x = \mp 2\sqrt{\frac{1}{3}p}$. But this would not be the only root. The second solution would give $\operatorname{cosec} 3A = 1$: therefore $A = \frac{90^\circ}{3}$, and then

$$x = \pm \sin 30^\circ \cdot 2\sqrt{\tfrac{1}{3}p} = \pm \sqrt{\tfrac{1}{3}p}.$$
$$x = \pm \sin 30^\circ \cdot 2\sqrt{\tfrac{1}{3}p} = \pm \sqrt{\tfrac{1}{3}p}.$$
$$x = \mp \sin 90^\circ \cdot 2\sqrt{\tfrac{1}{3}p} = \mp 2\sqrt{\tfrac{1}{3}p}.$$

Here it is obvious that the first two roots are equal, that their sum is equal to the third with a contrary sign, and that this third is the one which is produced from the first solution*.

In these solutions, the double signs in the value of x, relate to the double signs in the value of q.

N. B. Cardan's Rule for the solution of Cubics is given in the first volume of this course.

---

* The tables of sines, tangents, &c, besides their use in trigonometry, and in the solution of the equations, are also very useful in finding the value of algebraic expressions where extraction of roots would be otherwise required. Thus if a and b be any two quantities, of which a is the greater. Find x, z, &c, so, that $\tan x = \sqrt{\frac{b}{a}}$, $\sin z = \sqrt{\frac{b}{a}}$, $\sec y = \frac{a}{b}$, $\tan u = \frac{b}{a}$, and $\sin t = \frac{b}{a}$: then will

$$\log \sqrt{(a^2 - b^2)} = \log a + \log \sin y = \log b + \log \tan y.$$
$$\log \sqrt{(a^2 - b^2)} = \tfrac{1}{2}[\log (a + b) + \log (a - b)].$$
$$\log \sqrt{(a^2 + b^2)} = \log a + \log \sec u = \log b + \log \operatorname{cosec} u.$$
$$\log \sqrt{(a + b)} = \tfrac{1}{2} \log a + \log \sec x = \tfrac{1}{2} \log a + \tfrac{1}{2} \log 2 + \log \cos \tfrac{1}{2}y.$$
$$\log \sqrt{(a - b)} = \tfrac{1}{2} \log a + \log \cos z = \tfrac{1}{2} \log a + \tfrac{1}{2} \log 2 + \log \sin \tfrac{1}{2}y.$$
$$\log (a \pm b)^{\frac{m}{n}} = \frac{m}{n}[\log a + \log \cos t + \log \tan 45^\circ \pm \tfrac{1}{2}t].$$

The first three of these formulæ will often be useful, when two sides of a right-angled triangle are given, to find the third.

III. *Solution*

III. *Solution of Biquadratic Equations.*

Let the proposed biquadratic be $x^4 + 2px^3 = qx^2 + rx + s$. Now $(x^2 + px + n)^2 = x^4 + 2px^3 + (p^2 + 2n)x^2 + 2pnx + n^2$: if therefore $(p^2 + 2n)x^2 + 2pnx + n^2$ be added to both sides of the proposed biquadratic, the first will become a complete square $(x^2 + px + n)^2$, and the latter part $(p^2 + 2n + q)x^2 + (2pn + r)x + n^2 + s$, is a complete square if $4(p^2 + 2n + q) \,.\, (n^2 + s) = 2pn + r^2$; that is, multiplying and arranging the terms according to the dimensions of n, if $8n^3 + 4qn^2 + (8s - 4rp)n + 4qs + 4p^2s - r^2 = 0$. From this equation let a value of n be obtained, and substituted in the equation $(x^2 + px + n)^2 = (p^2 + 2n + q)x^2 + (2pn + r)x + n^2 + s$; then, extracting the square root on both sides

$$x^2+px+n=\pm\left\{\sqrt{(p^2+2n+q)}x+\sqrt{(n^2+s)}\right\}\begin{cases}\text{when } 2pn+r\\ \text{is positive;}\end{cases}$$

$$\text{or } x^2+px+n=\pm\left\{\sqrt{(p^2+2n+q)}x-\sqrt{(n^2+s)}\right\}\begin{cases}\text{when } 2pn+r\\ \text{is negative:}\end{cases}$$

And from these two quadratics, the four roots of the given biquadratic may be determined *.

Note. Whenever, by taking away the second term of a biquadratic, after the manner described in cor. th. 3, the fourth term also vanishes, the roots may immediately be obtained by the solution of a quadratic only.

A biquadratic may also be solved independently of cubics, in the following cases:

1. When the difference between the coefficient of the third term, and the square of half that of the second term, is equal to the coefficient of the fourth term, divided by half that of the second. Then if p be the coefficient of the second term, the equation will be reduced to a quadratic by dividing it by $x^2 \pm \frac{1}{2}px$.

2. When the last term is negative, and equal to the square of the coefficient of the fourth term divided by 4 times that of the third term, *minus* the square of that of the second: then to complete the square, subtract the terms of the proposed biquadratic from $(x^2 \pm \frac{1}{2}px)^2$, and add the remainder to both its sides.

3. When the coefficient of the fourth term divided by that of the second term, gives for a quotient the square root of the last term: then to complete the square, add the square of half the coefficient of the second term, to twice the square

* This rule, for solving biquadratics, by conceiving each to be the difference of two squares, is frequently ascribed to Dr. *Waring*; but its original inventor was Mr. *Thomas Simpson*, formerly Professor of Mathematics in the Royal Military Academy.

root

root of the last term, multiply the sum by x^2, from the product take the third term, and add the remainder to both sides of the biquadratics.

4. The fourth term will be made to go out by the usual operation for taking away the second term, when the difference between the cube of half the coefficient of the second term and half the product of the coefficients of the second and third term, is equal to the coefficient of the fourth term.

IV. *Euler's Rule for the Solution of Biquadratics.*

Let $x^4 - ax^2 - bx - c = 0$, be the given biquadratic equation wanting the second term. Take $f = \frac{1}{2}a$, $g = \frac{1}{16}a^2 + \frac{1}{4}c$, and $h = \frac{1}{64}b^2$, or $\sqrt{h} = \frac{1}{8}b$; with which values of f, g, h, form the cubic equation $z^3 - fz^2 + gz - h = 0$. Find the roots of this cubic equation, and let them be called p, q, r. Then shall the four roots of the proposed biquadratic be these following: viz.

When $\frac{1}{8}b$ is positive.	When $\frac{1}{8}b$ is negative:
1. $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$	$x = \sqrt{p} + \sqrt{q} - \sqrt{r}$.
2. $x = \sqrt{p} - \sqrt{q} - \sqrt{r}$.	$x = \sqrt{p} - \sqrt{q} + \sqrt{r}$.
3. $x = -\sqrt{p} + \sqrt{q} - \sqrt{r}$.	$x = -\sqrt{p} + \sqrt{q} + \sqrt{r}$.
4. $x = -\sqrt{p} - \sqrt{q} + \sqrt{r}$.	$x = -\sqrt{p} - \sqrt{q} - \sqrt{r}$.

Note 1. In any biquadratic equation having all its terms, if $\frac{3}{8}$ of the square of the coefficient of the 2d term be greater than the product of the coefficients of the 1st and 3d terms, or $\frac{3}{8}$ of the square of the coefficient of the 4th term be greater than the product of the coefficients of the 3d and 5th terms, or $\frac{4}{9}$ of the square of the coefficient of the 3d term greater than the product of the coefficients of the 2d and 4th terms; then all the roots of that equation will be real and unequal: but if either of the said parts of those squares be less than either of those products, the equation will have imaginary roots.

2. In a biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$, of which two roots are impossible, and d an affirmative quantity, then the two possible roots will be both negative, or both affirmative, according as $a^3 - 4ab + 8c$, is an affirmative or a negative quantity, if the signs of the coefficients a, b, c, d, are neither all affirmative, nor alternately $-$ and $+$ *.

* Various *general* rules for the solution of equations have been given by Demoivre, Bezout, Lagrange, &c; but the most universal in their application are approximating rules, of which a very simple and useful one is given in our first volume.

EXAMPLES.

EXAMPLES.

Ex. 1. Find the roots of the equation $x^2 + \frac{7}{44}x = \frac{1695}{12716}$, by tables of sines and tangents.

Here $p = \frac{7}{44}$, $q = \frac{1695}{12716}$, and the equation agrees with the 1st form. Also tan A $= \frac{88}{7}\sqrt{\frac{1695}{12716}}$, and $x = \tan \frac{1}{2}A = \sqrt{\frac{1695}{12716}}$.

In logarithms thus:

$$
\begin{array}{rl}
\text{Log } 1695 = & 3\cdot2291697 \\
\text{Arith. com. log } 12716 = & 5\cdot8956495 \\
\hline
\text{sum} + 10 = & 19\cdot1248192 \\
\hline
\text{half sum} = & 9\cdot5624096 \\
\log 88 = & 1\cdot9444827 \\
\text{Arith. com. log } 7 = & 9\cdot1549020 \\
\hline
\text{sum} - 10 = \log\tan A = & 10\cdot6617943 = \log\tan 77^\circ 42' 31''\tfrac{3}{4}; \\
\hline
\log\tan \tfrac{1}{2}A = & 9\cdot9061115 = \log\tan 38^\circ 51' 15''\tfrac{7}{8}; \\
\log\sqrt{q}, \text{ as above} = & 9\cdot5624096 \\
\hline
\text{sum} - 10 = \log x = - & 1\cdot4685211 = \log \cdot2941176.
\end{array}
$$

This value of x, viz ·2941176, is nearly equal to $\frac{5}{17}$. To find whether that is the exact root, take the arithmetical compliment of the last logarithm, viz 0·5314379, and consider it as the logarithm of the denominator of a fraction whose numerator is unity: thus is the fraction found to be $\frac{1}{3\cdot4}$ exactly, and this is manifestly equal to $\frac{5}{17}$. As to the other root of the equation, it is equal to $-\frac{1695}{12716} \div \frac{5}{17} = -\frac{339}{748}$.

Ex. 2. Find the roots of the cubic equation $x^3 - \frac{403}{441}x + \frac{46}{147} = 0$, by a table of sines.

Here $p = \frac{403}{441}$, $q = \frac{46}{147}$, the second term is negative, and $4p^3 > 27q^2$: so that the example falls under the irreducible case. Hence, $\sin 3A = \frac{3 \times 46}{147} \times \frac{441}{403} \times \frac{1}{2\sqrt{\frac{403}{3\cdot441}}} = \frac{414}{403} \cdot \frac{1}{\sqrt{\frac{1612}{1323}}}$.

The three values of x therefore, are

$$x = \sin A \sqrt{\frac{1612}{1323}}.$$

$$x = \sin(60^\circ - A)\sqrt{\frac{1612}{1323}}.$$

$$x = -\sin(60^\circ + A)\sqrt{\frac{1612}{1323}}.$$

The

The logarithmic computation is subjoined.

$$\begin{aligned}
\text{Log } 1612 &= 3{\cdot}2073650 \\
\text{Arith. com. log } 1323 &= 6{\cdot}8784402 \\
\text{sum} - 10 \ldots\ldots &= 0{\cdot}0858052 \\
\text{half sum} &= 0{\cdot}0429026 \text{ const. log.} \\
\text{Arith. com. const. log} &= 9{\cdot}9570974 \\
\log 414 \ldots &= 2{\cdot}6170003 \\
\text{Arith. com. log } 403 &= 7{\cdot}3946950 \\
\log \sin 3\text{A} \ldots &= 9{\cdot}9687927 = \log \sin 68^\circ\ 32'\ 18''\tfrac{1}{2}. \\
\text{Log } \sin \text{A} &= 9{\cdot}5891206 \\
\text{const. log} & 0{\cdot}0429026 \\
1.\ \text{sum} - 10 = \log x &= -\ \bar{1}{\cdot}6320232 = \log {\cdot}4285714 = \log \tfrac{3}{7}. \\
\text{Log } \sin (60^\circ - \text{A}) &= 9{\cdot}7810061 \\
\text{const log} \ldots\ldots &= 0{\cdot}0429026 \\
2.\ \text{sum} - 10 = \log x &= -\ \bar{1}{\cdot}8239087 = \log {\cdot}6666666 = \log \tfrac{2}{3}. \\
\text{Log } \sin (60^\circ + \text{A}) &= 9{\cdot}9966060 \\
\text{const. log} \ldots\ldots &= 0{\cdot}0429026 \\
3.\ \text{sum} - 10 = \log - x &= 0{\cdot}0395086 = \log 1{\cdot}095238 = \log \tfrac{23}{21}.
\end{aligned}$$

So that the three roots are $\frac{3}{7}$, $\frac{2}{3}$, and $-\frac{23}{21}$; of which the first two are together equal to the third with its sign changed, as they ought to be.

Ex. 3. Find the roots of the biquadratic $x^4 - 25x^2 + 60x - 36 = 0$, by Euler's rule.

Here $a = 25$, $b = -60$, and $c = 36$; therefore

$$f = \frac{25}{2},\ g = \frac{625}{16} + 9 = \frac{769}{16},\ \text{and } h = \frac{225}{4}.$$

Consequently the cubic equation will be

$$z^3 - \frac{25}{2}z^2 + \frac{769}{16}z - \frac{225}{4} = 0.$$

The three roots of which are

$$z = \frac{9}{4} = p, \text{ and } z = 4 = q, \text{ and } z = \frac{25}{4} = r;$$

the square roots of these are $\sqrt{p} = \frac{3}{2}$, $\sqrt{q} = 2$ or $\frac{4}{2}$, $\sqrt{r} = \frac{5}{2}$. Hence, as the value of $\frac{1}{8}b$ is negative, the four roots are

$$\begin{aligned}
\text{1st. } x &= \tfrac{3}{2} + \tfrac{4}{2} - \tfrac{5}{2} = 1, \\
\text{2d. } x &= \tfrac{3}{2} - \tfrac{4}{2} + \tfrac{5}{2} = 2, \\
\text{3d. } x &= -\tfrac{3}{2} + \tfrac{4}{2} + \tfrac{5}{2} = 3, \\
\text{4th. } x &= -\tfrac{3}{2} - \tfrac{4}{2} - \tfrac{5}{2} = -6.
\end{aligned}$$

Ex. 4. Produce a quadratic equation whose roots shall be $\frac{3}{4}$ and $\frac{4}{5}$. Ans. $x^2 - \frac{31}{20}x + \frac{3}{5} = 0$.

Ex. 5. Produce a cubic equation whose roots shall be 2, 5, and -3. Ans. $x^3 - 4x^2 - 11x + 30 = 0$.

Ex. 6.

Ex. 6. Produce a biquadratic which shall have for the roots 1, 4, − 5, and 6 respectively.

Ans. $x^4 - 6x^3 - 21x^2 + 146x - 120 = 0$.

Ex. 7. Find x, when $x^2 + 347x = 22110$.

Ans. $x = 55$, $x = - 402$.

Ex. 8. Find the roots of the quadratic $x^2 - \frac{55}{12}x - \frac{325}{6}$.

Ans. $x = 10$, $x = -\frac{65}{12}$.

Ex. 9. Solve the equation $x^2 - \frac{264}{25}x = -\frac{695}{25}$.

Ans. $x = 5$, $x = \frac{139}{25}$.

Ex. 10. Given $x^2 - 24113x = - 481860$, to find x.

Ans. $x = 20$, $x = 24093$.

Ex. 11. Find the roots of the equation $x^3 - 3x - 1 = 0$.
Ans. the roots are sine 70°, − sin 50°, and − sin 10°, to a radius = 2; or the roots are twice the sines of those arcs as given in the tables.

Ex. 12. Find the real root of $x^3 - x - 6 = 0$.

Ans. $\frac{2}{3}\sqrt{3} \times \sec 54°44'20''$.

Ex. 13. Find the real root of $25x^3 + 75x - 46 = 0$.

Ans. $2 \cot 74°27'48''$.

Ex. 14. Given $x^4 - 8x^3 - 12x^2 + 84x - 63 = 0$, to find x by quadratics. Ans. $x = 2 + \sqrt{7} \pm \sqrt{11 + \sqrt{7}}$.

Ex. 15. Given $x^4 + 36x^3 - 400x^2 - 3168x + 7744 = 0$, to find x, by quadratics. Ans. $x = 11 + \sqrt{209}$.

Ex. 16. Given $x^4 + 24x^3 - 114x^2 - 24x + 1 = 0$, to find x. Ans. $x = \pm\sqrt{197} - 14$, $x = 2 \pm \sqrt{5}$.

Ex. 17. Find x, when $x^4 - 12x - 5 = 0$.

Ans. $x = 1 \pm \sqrt{2}$, $x = - 1 \pm 2\sqrt{-1}$.

Ex. 18. Find x, when $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$.

Ans. $x = 1$, or 2, or 3, or 6.

Ex. 19. Given $x^5 - 5ax^4 - 80a^2x^3 - 68a^3x^2 + 7a^4x + a^5 = 0$, to find x.

Ans. $x = - a$, $x = 6a \pm a\sqrt{37}$, $x = \pm a\sqrt{10} - 3a$.

CHAPTER

CHAPTER IX.

ON THE NATURE AND PROPERTIES OF CURVES, AND THE CONSTRUCTION OF EQUATIONS.

SECTION I.

Nature and Properties of Curves.

DEF. 1. A curve is a line whose several parts proceed in different directions, and are successively posited towards different points in space, which also may be cut by one right line in two or more points.

If all the points in the curve may be included in one plane, the curve is called a *plane* curve; but if they cannot all be comprized in one plane, then is the curve one of *double curvature*.

Since the word direction implies straight lines, and in strictness no part of a curve is a right line, some geometers prefer defining curves otherwise: thus, in a straight line, to be called the line of the abscissas, from a certain point let a line arbitrarily taken be called the abscissa, and denoted (commonly) by x: at the several points corresponding to the different values of x, let straight lines be continually drawn, making a certain angle with the line of the abscissas: these straight lines being regulated in length according to a certain law or equation, are called ordinates; and the line or figure in which their extremities are continually found is, in general, a curve line. This definition however is not free from objection; for a right line may be denoted by an equation between its abscissas and ordinates, such as $y = ax + b$.

Curves are distinguished into algebraical or geometrical, and transcendental or mechanical.

Def. 2. *Algebraical* or geometrical curves, are those in which the relations of the abscissas to the ordinates can be denoted by a common algebraical expression: such, for example, as the equations to the conic sections, given in the first chapter of this volume.

Def. 3. *Transcendental* or mechanical curves, are such as cannot be so defined or expressed by a pure algebraical equation; or when they are expressed by an equation, having one of

of its terms a variable quantity, or a curve line. Thus, $y = \log x$, $y = \text{A} . \sin x$, $y = \text{A} . \cos x$, $y = \text{A}^x$, are equations to transcendental curves; and the latter in particular is an equation to an *exponential* curve.

Def. 4. Curves that turn round a fixed point or centre, gradually receding from it, are called *spiral* or *radial* curves.

Def. 5. *Family* or *tribe* of curves, is an assemblage of several curves of different kinds, all defined by the same equation of an indeterminate degree; but differently, according to the diversity of their kind. For example, suppose an equation of an indeterminate degree, $a^{m-1}x = y^m$: if $m = 2$, then will $ax = y^2$; if $m = 3$, then will $a^2x = y^3$; if $m = 4$, then is $a^3x = y^4$; &c: all which curves are said to be of the same family or tribe.

Def. 6. The *axis* of a figure is a right line passing through the centre of a curve, when it has one: if it bisects the ordinates, it is called a *diameter*.

Def. 7. An *asymptote* is a right line which continually approaches towards a curve, but never can touch it, unless the curve could be extended to an infinite distance.

Def. 8. An abscissa and an ordinate, whether right or oblique, are, when spoken of together, frequently termed *co-ordinates*.

ART. 1. The most convenient mode of classing algebraical curves, is according to the orders or dimensions of the equations which express the relation between the co-ordinates. For then the equation for the same curve, remaining always of the same order so long as each of the assumed systems of co-ordinates is supposed to retain constantly the same inclination of ordinate to abscissa, while referred to different points of the curve, however the axis and the origin of the abscissas, or even the inclination of the co-ordinates in different systems, may vary; the same curve will never be ranked under different orders, according to this method. If therefore we take, for a distinctive character, the number of dimensions which the co-ordinates, whether rectangular or oblique, form in the equation, we shall not disturb the order of the classes, by changing the axis and the origin of the abscissas, or by varying the inclination of the co-ordinates.

2. As algebraists call orders of different kinds of equations, those which constitute the greater or less number of dimensions, they distinguish by the same name the different kinds of resulting lines. Consequently the general equation of the first order being $0 = \alpha + \beta x + \gamma y$; we may refer to the

first

first order all the lines which, by taking x and y for the co-ordinates, whether rectangular or oblique, give rise to this equation. But this equation comprises the right line alone, which is the most simple of all lines; and since, for this reason, the name of curve does not properly apply to the first order, we do not usually distinguish the different orders by the name of curve lines, but simply by the generic term of lines: hence the first order of lines does not comprehend any curves, but solely the right line.

As for the rest, it is indifferent whether the co-ordinates are perpendicular or not; for if the ordinates make with the axis an angle ϕ whose sine is μ and cosine ν, we can refer the equation to that of the rectangular co-ordinates, by making $y = \frac{u}{\mu}$, and $x = \frac{\nu u}{\mu} + t$; which will give for an equation between the perpendiculars t and u,

$$0 = \alpha + \beta t + \left(\frac{\beta\nu}{\mu} + \frac{\gamma}{\mu}\right)u.$$

Thus it follows evidently, that the signification of the equation is not limited by supposing the ordinates to be rightly applied: and it will be the same with equations of superior orders, which will not be less general though the co-ordinates are perpendicular. Hence, since the determination of the inclination of the ordinates applied to the axis, takes nothing from the generality of a general equation of any order whatever, we put no restriction on its signification by supposing the co-ordinates rectangular; and the equation will be of the same order whether the co-ordinates be rectangular or oblique.

3. All the lines of the second order will be comprised in the general equation

$$0 = \alpha + \beta x + \gamma y + \delta x^2 + \varepsilon xy + \zeta y^2;$$

that is to say, we may class among lines of the second order all the curve lines which this equation expresses, x and y denoting the rectangular co-ordinates. These curve lines are therefore the most simple of all, since there are no curves in the first order of lines; it is for this reason that some writers call them curves of the first order. But the curves included in this equation are better known under the name of CONIC SECTIONS, because they all result from sections of the cone. The different kinds of these lines are the ellipse, the circle, or ellipse with equal axes, the parabola, and the hyperbola; the properties of all which may be deduced with facility from the preceding general equation. Or this equation may be transformed into the subjoined one:

$$y^2 + \frac{\varepsilon x + \gamma}{\zeta} y + \frac{\delta x^2 + \beta x + \alpha}{\zeta} = 0:$$

and

and this again may be reduced to the still more simple form $y^2 = fx^2 + gx + h$.

Here, when the first term fx^2 is *affirmative*, the curve expressed by the equation is a hyperbola; when fx^2 is *negative*, the curve is an ellipse: when that term is *absent*, the curve is a parabola. When x is taken upon a *diameter*, the equations reduce to those already given in sect. 4 ch. i.

The mode of effecting these transformations is omitted for the sake of brevity. This section contains a *summary*, not an *investigation* of properties: the latter would require many volumes, instead of a section.

4. Under lines of the third order, or curves of the second, are classed all those which may be expressed by the equation
$$0 = \alpha + \beta x + \gamma y + \delta x^2 + \epsilon xy + \zeta y^2 + \eta x^3 + \theta x^2 y + \iota xy^2 + \kappa y^3.$$
And in like manner we regard as lines of the fourth order, those curves which are furnished by the general equation
$$0 = \alpha + \beta x + \gamma y + \delta x^2 + \epsilon xy + \zeta y^2 + \eta x^3 + \theta x^2 y + \iota xy^2 + \kappa y^3 + \lambda x^4 + \mu x^3 y + \nu x^2 y^2 + \xi xy^3 + o y^4;$$
taking always x and y for rectangular co-ordinates. In the most general equation of the third order, there are 10 constant quantities, and in that of the fourth order 15, which may be determined at pleasure; whence it results that the kinds of lines of the third order, and, much more those of the fourth order, are considerably more numerous than those of the second.

5. It will now be easy to conceive, from what has gone before, what are the curve lines that appertain to the fifth, sixth, seventh, or any higher order; but as it is necessary to add to the general equation of the fourth order, the terms
$$x^5,\ x^4 y,\ x^3 y^2,\ x^2 y^3,\ xy^4,\ y^5,$$
with their respective constant coefficients, to have the general equation comprising all the lines of the fifth order, this latter will be composed of 21 terms: and the general equation comprehending all the lines of the sixth order, will have 28 terms; and so on, conformably to the law of the triangular numbers. Thus, the most general equation for lines of the order n, will contain $\frac{(n+1) \cdot (n+2)}{1 \cdot 2}$ terms, and as many constant letters, which may be determined at pleasure.

6. Since the order of the proposed equation between the co-ordinates, makes known that of the curve line; whenever we have given an algebraic equation between the co-ordinates x and y, or t and u, we know at once to what order it is necessary to refer the curve represented by that equation. If the equation be irrational, it must be freed from radicals, and if

if there be fractions, they must be made to disappear; this done, the greatest number of dimensions formed by the variable quantities x and y, will indicate the order to which the line belongs. Thus, the curve which is denoted by this equation $y^2 - ax = 0$, will be of the second order of lines, or of the first order of curves; while the curve represented by the equation $y^2 = x\sqrt{(a^2 - x^2)}$, will be of the third order (that is, the fourth order of lines), because the equation is of the fourth order when freed from radicals; and the line which is indicated by the equation $y = \frac{a^3 - ax^2}{a^2 + x^2}$, will be of the third order, or of the second order of curves, because the equation when the fraction is made to disappear, becomes $a^2y + x^2y = a^3 - ax^2$, where the term x^2y contains three dimensions.

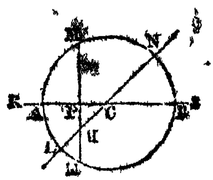
7. It is possible that one and the same equation may give different curves, according as the applicates or ordinates fall upon the axis perpendicularly or under a given obliquity. For instance, this equation, $y^2 = ax - x^2$, gives a circle, when the co-ordinates are supposed perpendicular; but when the co-ordinates are oblique, the curve represented by the same equation will be an ellipse. Yet all these different curves appertain to the same order, because the transformation of rectangular into oblique co-ordinates, and the contrary, does not affect the order of the curve, or of its equation. Hence, though the magnitude of the angles which the ordinates form with the axis, neither augments nor diminishes the generality of the equation, which expresses the lines of each order; yet, a particular equation being given, the curve which it expresses can only be determined when the angle between the co-ordinates is determined also.

8. That a curve line may relate properly to the order indicated by the equation, it is requisite that this equation be not decomposable into rational factors; for if it could be composed of two or of more such factors, it would then comprehend as many equations, each of which would generate a particular line, and the re-union of these lines would be all that the equation proposed could represent. Those equations, then, which may be decomposed into such factors, do not comprise one continued curve, but several at once, each of which may be expressed by a particular equation; and such combinations of separate curves are denoted by the term complex curves.

Thus, the equation $y^2 = ay + xy - ax$, which seems to appertain to a line of the second order, if it be reduced to zero by making $y^2 - ay - xy + ax = 0$, will be composed of the factors $(y - x)(y - a) = 0$; it therefore comprises the

the two equations $y - x = 0$, and $y - a = 0$, both of which belong to the right line: the first forms with the axis at the origin of the abscissas an angle equal to half a right angle; and the second is parallel to the axis, and drawn at a distance $= a$. These two lines, considered together, are comprized in the proposed equation $y^2 = ay + xy - ax$. In like manner we may regard as complex this equation, $y^4 - xy^3 - a^2x^2 - ay^3 + ax^2y + a^2xy = 0$; for its factors being $(y - x)(y - a)(y^2 - ax) = 0$, instead of denoting one continued line of the fourth order, it comprises three distinct lines, viz, two right lines, and one curve denoted by the equa. $y^2 - ax = 0$.

9. We may therefore form at pleasure any complex lines whatever, which shall contain 2 or more right lines or curves. For, if the nature of each line is expressed by an equation referred to the same axis, and to the same origin of the abscissas, and after having reduced each equation to zero, we multiply them one by another, there will result a complex equation which at once comprizes all the lines assumed. For example, if from the centre C, with a radius CA $= a$, a circle be described; and further, if a right line LN be drawn through the centre C; then we may, for any assumed axis, find an equation which will at once include the circle and the right line, as though these two lines formed only one.



Suppose there be taken for an axis the diameter AB, that forms with the right line LN an angle equal to half a right angle: having placed the origin of the abscissas in A, make the abscissa AP $= x$, and the applicate or ordinate PM $= y$; we shall have for the right line, PM $=$ CP $= a - x$; and since the point M of the right line falls on the side of those ordinates which are reckoned negative, we have $y = -a + x$, or $y - x + a = 0$: but, for the circle, we have PM^2 $=$ AP . PB, and BP $= 2a - x$, which gives $y^2 = 2ax - x^2$, or $y^2 + x^2 - 2ax = 0$. Multiplying these two equations together we obtain the complex equation of the third order,

$$y^3 - y^2x + yx^2 - x^3 + ay^2 - 2axy + 3ax^2 - 2a^2x = 0,$$

which represents, at once, the circle and the right line. Hence, we shall find that to the abscissa AP $= x$, corresponds three ordinates, namely, two for the circle, and one for the right line. Let, for example, $x = \frac{1}{2}a$, the equation will become $y^3 + \frac{1}{2}ay^2 - \frac{3}{4}a^2y - \frac{3}{8}a^3 = 0$; whence we first find $y + \frac{1}{2}a = 0$, and by dividing by this root, we obtain $y^2 - \frac{3}{4}a^2 = 0$, the two roots of which being taken and ranked with the former, give the three following values of y:

 I. $y =$

I. $y = -\frac{1}{2}a$.
II. $y = +\frac{1}{2}a\sqrt{3}$.
III. $y = -\frac{1}{2}a\sqrt{3}$.

We see, therefore, that the whole is represented by one equation, as if the circle together with the right line formed only one continued curve.

10. This difference between simple and complex curves being once established, it is manifest that the lines of the second order are either continued curves, or complex lines formed of two right lines; for if the general equation have rational factors, they must be of the first order, and consequently will denote right lines. Lines of the third order will be either simple, or complex, formed either of a right line and a line of the second order, or of three right lines. In like manner, lines of the fourth order will be continued and simple, or complex, comprizing a right line and a line of the third order, or two lines of the second order, or lastly, four right lines. Complex lines of the fifth and superior orders will be susceptible of an analogous combination, and of a similar enumeration. Hence it follows, that any order whatever of lines may comprize, at once, all the lines of inferior order, that is to say, that they may contain a complex line of any inferior orders with one or more right lines, or with lines of the second, third, &c, order; so that if we sum the numbers of each order, appertaining to the simple lines, there will result the number indicating the order of the complex line.

Def. 9. That is called an *hyperbolic* leg, or branch of a curve, which approaches constantly to some asymptote; and that a *parabolic* one which has no asymptote.

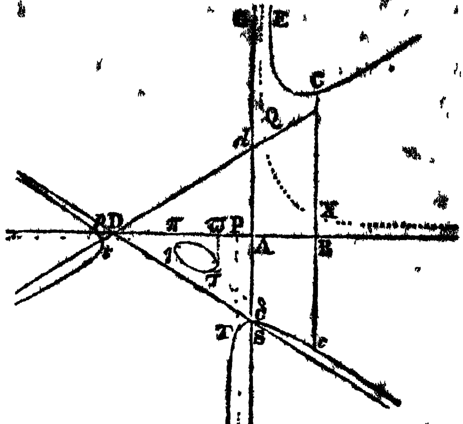
Art. 11. All the legs of curves of the second and higher kinds, as well as of the first, infinitely drawn out, will be of either the hyperbolic or the parabolic kind: and these legs are best known from the tangents. For if the point of contact be at an infinite distance, the tangent of a hyperbolic leg will coincide with the asymptote, and the tangent of a parabolic leg will recede *in infinitum*, will vanish and be no where found. Therefore the asymptote of any leg is found by seeking the tangent to that leg at a point infinitely distant: and the course, or way of an infinite leg, is found by seeking the position of any right line which is parallel to the tangent where the point of contact goes off *in infinitum*: for this right line is directed the same way with the infinite leg.

Sir Isaac Newton's Reduction of all Lines of the Third Order, to four Cases of Equations; with the Enumeration of those Lines.

CASE I.

CASE I.

12. All the lines of the first, third, fifth, and seventh order, or of any odd order, have at least two legs or sides proceeding on *ad infinitum*, and towards contrary parts. And all lines of the *third* order have two such legs or branches running out contrary ways, and towards which no other of their infinite legs (except in the Cartesian parabola) tend. If the legs are of the *hyperbolic* kind, let GAS be their asymptote; and to it



let the parallel CB*c* be drawn, terminated (if possible) at both ends at the curve. Let this parallel be bisected in X, and then will the locus of that point X be the conical or common hyperbola XQ, one of whose asymptotes is AS. Let its other asymptote be AB. Then the equation by which the relation between the ordinate BC $= y$, and the abscissa AB $= x$, is determined, will always be of this form: viz,

$$xy^2 + ey = ax^3 + bx^2 + cx + d \ldots \text{(I.)}$$

Here the coefficients e, a, b, c, d, denote given quantities, affected with their signs $+$ and $-$, of which terms any one may be wanting, provided the figure through their defect does not become transformed into a conic section. The conical hyperbola XQ may coincide with its asymptotes, that is, the point X may come to be in the line AB; and then the term $+ ey$ will be wanting.

CASE II.

13. But if the right line CB*c* cannot be terminated both ways at the curve, but will come to it only in one point; then draw any line in a given position which shall cut the asymptote AS in A; as also any other right line, as BC, parallel to the

the asymptote, and meeting the curve in the point C, then the equation, by which the relation between the ordinate BC and the abscissa AB is determined, will always assume this form: viz. $xy = ax^3 + bx^2 + cx + d$ . . . . (II.)

CASE III.

14. If the opposite legs be of the parabolic kind, draw the right line CBC, terminated at both ends (if possible) at the curve, and running according to the course of the legs; which line bisect in B: then shall the locus of B be a right line. Let that right line be AB, terminated at any given point, as A: then the equation, by which the relation between the ordinate BC and the abscissa AB is determined, will always be of this form: $y^2 = ax^3 + bx^2 + cx + d$ . . . . (III.)

CASE IV.

15. If the right line CBC meet the curve only in one point, and therefore cannot be terminated at the curve at both ends; let the point where it comes to the curve be C, and let that right line at the point B, fall on any other right line given in position, as AB, and terminated at any given point, as A. Then will the equation expressing the relation between BC and AB, assume this form:

$$y = ax^3 + bx^2 + cx + d \; . \; . \; . \; . \; \text{(IV.)}$$

16. In the first case, or that of equation I, if the term ax^3 be affirmative, the figure will be a triple hyperbola with six hyperbolic legs, which will run on infinitely by the three asymptotes, of which none are parallel, two legs towards each asymptote, and towards contrary parts; and these asymptotes, if the term bx^2 be not wanting in the equation, will mutually intersect each other in 3 points, forming thereby the triangle $Dd\delta$. But if the term bx^2 be wanting, they will all converge to the same point. This kind of hyperbola is called *redundant*, because it exceeds the conic hyperbola in the number of its hyperbolic legs.

In every redundant hyperbola, if neither the term ey be wanting, nor $b^2 - 4ac = ae\sqrt{a}$, the curve will have *no* diameter; but if either of those occur separately, it will have only *one* diameter; and *three*, if they both happen. Such diameter will always pass through the intersection of two of the asymptotes, and bisect all right lines which are terminated each way by those asymptotes, and which are parallel to the third asymptote.

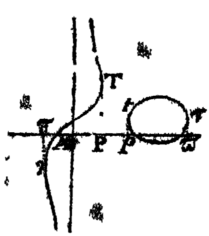
17. If the redundant hyperbola have no diameter, let the four roots or values of x in the equation $ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}e^2 = 0$, be sought; and suppose them to be AP, Aπ,

Aπ,

Aπ, and Ap (see the preceding figure). Let the ordinates PT, ϖτ, πt, pt, be erected; they shall touch the curve in the points T, τ, t, t, and by that contact shall give the limits of the curve, by which its species will be discovered.

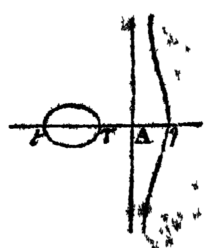
Thus, if all the roots AP, Aϖ, Aπ, Ap, be real, and have the same sign, and are unequal, the curve will consist of three hyperbolas and an oval; viz, an *inscribed hyperbola* as AC; a *circumscribed hyperbola*, as Tδc; an *ambigeneal hyperbola*, (i. e. lying within one asymptote and beyond another) as pt; and an oval τt. This is reckoned the *first* species. Other relations of the roots of the equation, give 8 more different species of redundant hyperbolas without diameters; 12 each with but *one* diameter; 2 each with *three* diameters; and 9 each with three asymptotes converging to a common point. Some of these have ovals, some points of decussation, and in some the ovals degenerate into nodes or knots.

18. When the term ax^3 in equa. 1, is negative, the figure expressed by that equation, will be a deficient or *defective hyperbola*, that is, it will have fewer legs than the complete conic hyperbola. Such is the marginal figure, representing Newton's 33d species; which is constituted of an *anguineal* or serpentine hyperbola, (both legs approaching a common asymptote by means of a contrary flexure,) and a conjugate oval. There are 6 species of defective hyperbolas, each having but one asymptote, and only two hyperbolic legs, running out contrary ways, *ad infinitum*; the asymptote being the first and principal ordinate.



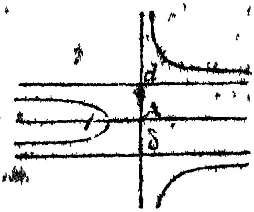
When the term ey is not absent, the figure will have no diameter; when it is absent, the figure will have one diameter. Of this latter class there are 7 different species, one of which, namely Newton's 40th species, is exhibited in the margin.

19. If, in equation 1, the term ax^3 be wanting, but bx^2 not, the figure expressed by the equation remaining, will be a parabolic hyperbola, having two hyperbolic legs to one asymptote, and two parabolic legs converging one and the same way. When the term ey is not wanting, the figure will have no diameter; if that term be wanting, the figure will have one diameter. There are 7 species appertaining to the former case; and 4 to the latter.



20. When,

20. When, in equa. I, the terms ax^3, bx^2, are wanting, or when that equation becomes $xy^2 + ey = cx + d$, it expresses a figure consisting of three hyperbolas opposite to one another, one lying between the parallel asymptotes, and the other two without: each of these curves having three asymptotes, one of which is the first and principal ordinate, the other two parallel to the abscissa, and equally distant from it, as in the annexed figure of Newton's 60th species. Otherwise the said equation expresses two opposite circumscribed hyperbolas, and an anguineal hyperbola between the asymptotes. Under this class there are 4 species, called by Newton *Hyperbolismæ of an hyperbola.* By hyperbolismæ of a figure he means to signify when the ordinate comes out, by dividing the rectangle under the ordinate of a given conic section and a given right line, by the common abscissa.

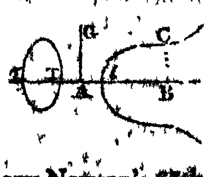


21. When the term cx^2 is negative, the figure expressed by the equation $xy^2 + ey = - cx^2 + d$, is either a serpentine hyperbola, having only one asymptote, being the principal ordinate; or else it is a conchoidal figure. Under this class there are 3 species, called *Hyperbolismæ of an ellipse.*

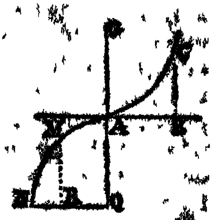
22. When the term cx^2 is absent, the equa. $xy^2 + ey = d$, expresses two hyperbolas, lying, not in the opposite angles of the asymptotes (as in the conic hyperbola), but in the adjacent angles. Here there are only 2 species, one consisting of an inscribed and an ambigeneal hyperbola, the other of two inscribed hyperbolas. These two species are called the *Hyperbolismæ of a parabola.*

23. In the second case of equations, or that of equation II, there is but one figure; which has four infinite legs. Of these, two are hyperbolic about one asymptote, tending towards contrary parts, and two converging parabolic legs, making with the former nearly the figure of a *trident*, the familiar name given to this species. This is the *Cartesian* parabola, by which equations of 6 dimensions are sometimes constructed: it is the 66th species of Newton's enumeration.

24. The third case of equations, or equa. III, expresses a figure having two parabolic legs running out contrary ways: of these there are 5 different species, called *diverging* or *bell-form* parabolas; of which 2 have ovals, 1 is nodate, 1 punctate, and 1 cuspidate. The figure shows Newton's 67th species;



species; in which the oval must always be so small that no right line which cuts it twice can cut the parabolic curve at more than once.



25. In the case to which equa. IV. refers, there is but one species. It expresses the *cubical* parabola with contrary legs. This curve may easily be described mechanically by means of a square and an equilateral hyperbola. Its most simple property is, that RM (parallel to AQ) always varies as $QN^3 - QR^3$.

26. Thus according to Newton there are 72 species of lines of the third order. But Mr. Stirling discovered four more species of redundant hyperbolas; and Mr. Stone two more species of deficient hyperbolas, expressed by the equation $xy^2 = bx^2 + cx + d$: i. e. in the case when $bx^2 + cx + d = 0$, has two unequal negative roots, and in that where the equation has two equal negative roots. So that there are *at least* 78 different species of lines of the third order. Indeed Euler, who classes all the varieties of lines of the third order under 16 general species, affirms that they comprehend more than 80 varieties; of which the preceding enumeration necessarily comprizes nearly the whole.

27. Lines of the fourth order are divided by Euler into 146 classes; and these comprize more than 5000 varieties: they all flow from the different relations of the quantities in the 10 general equations subjoined.

$$\left.\begin{array}{ll}
1. & y^4 + fx^2y^2 + gxy^3 + hx^2y + iy^2 + kxy + ly \\
2. & y^4 + fxy^2 + gx^2y + hxy^2 + ixy + ky \;.\;. \\
3. & x^2y^2 + fy^3 + gx^2y + hy^2 + ly \;.\;.\;.\;. \\
4. & x^2y^2 + fy^3 + gx^2 + hxy + iy \;.\;.\;.\;. \\
5. & y^3 + fxy^2 + gx^2y + hy \;.\;.\;.\;.\;.\;. \\
6. & y^3 + fxy^2 + gxy + hy \;.\;.\;.\;.\;.\;.
\end{array}\right\} = ax^4 + bx^3 + cx^2 + dx + e$$

$$\left.\begin{array}{ll}
7. & y^4 + ex^3y + fxy^3 + gxy^2 + hy^2 + ixy + ky \\
8. & x^3y + exy^3 + fxy + gy^2 + hxy + iy \;.\;. \\
9. & x^3y + ey^3 + fxy^2 + gxy + hy \;.\;.\;.\;. \\
10. & x^3y + ey^3 + fy^2 + gxy + hy \;.\;.
\end{array}\right\} = ax^3 + bx^2 + cx + d.$$

28. Lines of the fifth and higher orders, of necessity become still more numerous; and present too many varieties to admit of any classification, at least in moderate compass. Instead, therefore, of dwelling upon these; we shall give a concise sketch of the most curious and important properties of curve lines in general, as they have been deduced from a contemplation of the nature and mutual relation of the roots of the equations representing those curves. Thus a curve being called of n dimensions, or a line of the nth order when its representative equation rises to n dimensions; then since

for

for every different value of x there are n values of y, it will commonly happen that the ordinate will cut the curve in n or in $n-2$, $n-4$, &c, points, according as the equation has n, or $n-2$, $n-4$, &c, possible roots. It is not however to be inferred, that a right line will cut a curve of n dimensions, in n, $n-2$, $n-4$, &c, points, only; for if this were the case, a line of the 2d order, a conic section for instance, could only be cut by a right line in two points;—but this is manifestly incorrect, for though a conic parabola will be cut in two points by a right line oblique to the axis, yet a right line parallel to the axis can only cut the curve in one point.

29. It is true in general, that lines of the n order cannot be cut by a right line in more than n points; but it does not hence follow, that any right line whatever will cut in n points every line of that order; it may happen that the number of intersections is $n-1$, $n-2$, $n-3$, &c, to $n-n$. The number of intersections that any right line whatever makes with a given curve line, cannot therefore determine the order to which a curve line appertains. For, as Euler remarks, if the number of intersections be $= n$, it does not follow that the curve belongs to the n order, but it may be referred to some superior order; indeed it may happen that the curve is not algebraic, but transcendental. This case excepted, however, Euler contends that we may always affirm positively that a curve line which is cut by a right line in n points, cannot belong to an order of lines inferior to n. Thus, when a right line cuts a curve in 4 points, it is certain that the curve does not belong to either the second or third order of lines; but whether it be referred to the fourth, or a superior order, or whether it be transcendental, is not to be decided by analysis.

30. Dr. Waring has carried this enquiry a step further than Euler, and has demonstrated that there are curves of any number of odd orders, that cut a right line in 2, 4, 6, &c, points only; and of any number of even orders that cut a right line in 3, 5, 7, &c, points only; whence this author likewise infers, that the order of the curve cannot be announced from the number of points in which it cuts a right line. See his Proprietates Algebraicarum Curvarum.

31. Every geometrical curve being continued, either returns into itself, or goes on to an infinite distance. And if any plane curve has two infinite branches or legs, they join one another either at a finite, or at an infinite distance.

32. In any curve, if tangents be drawn to all points of the curve; and if they always cut the abscissa at a finite distance from its origin; that curve has an asymptote, otherwise not.

33. A

33. A line of any order may have as many asymptotes as it has dimensions, and no more.

34. An asymptote may intersect the curve in so many points abating two, as the equation of the curve has dimensions. Thus, in a conic section, which is the second order of lines, the asymptote does not cut the curve at all; in the third order it can only cut it in one point; in the fourth order, in two points; and so on.

35. If a curve have as many asymptotes, as it has dimensions, and a right line be drawn to cut them all, the parts of that measured from the asymptotes to the curve, will together be equal to the parts measured in the same direction, from the curve to the asymptotes.

36. If a curve of n dimensions have n asymptotes, then the content of the n abscissas will be to the content of the n ordinates, in the same ratio in the curve and asymptotes; the sum of their n subnormals, to ordinates perpendicular to their abscissas, will be equal to the curve and the asymptotes; and they will have the same central and diametral curves.

37. If two curves of n and m dimensions have a common asymptote; or the terms of the equations to the curves of the greatest dimensions have a common divisor; then the curves cannot intersect each other in $n \times m$ points, possible or impossible. If the two curves have a common general centre, and intersect each other in $n \times m$ points, then the sum of the affirmative abscissas, &c, to those points, will be equal to the sum of the negative; and the sum of the n subnormals to a curve which has a general centre, will be proportional to the distance from that centre.

38. Lines of the third, fifth, seventh, &c order, or any odd number, have, as before remarked, at least two infinite legs or branches, running contrary ways; while in lines of the second, fourth, sixth, or any even number of dimensions, the figure may return into itself, and be contained within certain limits.

39. If the right lines AP, PM, forming a given angle APM, cut a geometrical line of any order in as many points, as it has dimensions, the product of the segments of the first terminated by P and the curve, will always be to the product of the segments of the latter, terminated by the same point and the curve, in an invariable ratio.

40. With respect to double, triple, quadruple, and other multiple points, or the points of intersection of 2, 3, 4, or more branches of a curve, their nature and number may be estimated by means of the following principles. 1. A curve of the n order is determinate when it is subjected to pass through the

the number $\frac{(n+1).(n+2)}{2} - 1$ points. 2. A curve of the n order cannot intersect a curve of the m order in more than mn points.

Hence it follows that a curve of the second order, for example, can always pass through 5 given points (not in the same right line), and cannot meet a curve of the m order in more than mn points; and it is impossible that a curve of the m order should have 5 points whose degrees of multiplicity make together more than $2m$ points. Thus, a line of the fourth order cannot have four double points; because the line of the second order which would pass through these four double points, and through a fifth simple point of the curve of the fourth dimension, would meet 9 times; which is impossible, since there can only be an intersection 2×4 or 8 times.

For the same reason, a curve line of the 5th order cannot, with one triple point, have more than three double points: and in a similar manner we may reason for curves of higher orders.

Again, from the known proposition, that we can always make a line of the third order pass through nine points, and that a curve of that order cannot meet a curve of the m order in more than $3m$ points, we may conclude that a curve of the m order cannot have nine points, the degrees of multiplicity of which make together a number greater than $3m$. Thus, a line of the fifth order cannot have more than 6 double points; a line of the 6th order, which cannot have more than one quadruple point, cannot have with that quadruple point more than 6 double points; nor with two triple points more than 5 double points; nor even with one triple point more than 7 double points. Analogous conclusions obtain with respect to a line of the fourth order, which we may cause to pass through 14 points, and which can only meet a curve of the m order in $4m$ points, and so on.

41. The properties of curves of a superior order, agree, under certain modifications, with those of all inferior orders. For though some line or lines become evanescent, and others become infinite, some coincide, others become equal; some points coincide, and others are removed to an infinite distance; yet, under these circumstances, the general properties still hold good with regard to the remaining quantities; so that whatever is demonstrated generally of any order, holds true in the inferior orders: and, on the contrary, there is hardly any property of the inferior orders, but there is some similar to it, in the superior ones.

For, as in the conic sections, if two parallel lines are drawn

to

to terminate at the section, the right line that bisects these will bisect all other lines parallel to them; and is therefore called a *diameter* of the figure, and the bisected lines *ordinates*, and the intersections of the diameter with the curve *vertices*; the common intersection of all the diameters the *centre*; and that diameter which is perpendicular to the ordinates, the *vertex*. So likewise in higher curves, if two parallel lines be drawn, each to cut the curve in the number of points that indicate the order of the curve; the right line that cuts these parallels so, that the sum of the parts on one side of the line, estimated to the curve, is equal to the sum of the parts on the other side, it will cut in the same manner all other lines parallel to them that meet the curve in the same number of points; in this case also the divided lines are called *ordinates*, the line so dividing them a *diameter*, the intersection of the diameter and the curve *vertices*; the common intersection of two or more diameters the *centre*; the diameter perpendicular to the ordinates, if there be any such, the *axis*; and when all the diameters concur in one point, that is the *general centre*.

Again, the conic hyperbola, being a line of the second order, has two asymptotes; so likewise, that of the third order may have three; that of the fourth, four: and so on: and they can have no more. And as the parts of any right line between the hyperbola and its asymptotes are equal; so likewise in the third order of lines, if any line be drawn cutting the curve and its asymptotes in three points; the sum of two parts of it falling the same way from the asymptotes to the curve, will be equal to the part falling the contrary way from the third asymptote to the curve; and so of higher curves.

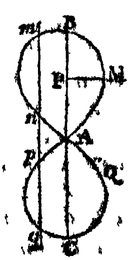
Also, in the conic sections which are not parabolic: as the square of the ordinate, or the rectangle of the parts of it on each side of the diameter, is to the rectangle of the parts of the diameter, terminating at the vertices, in a constant ratio, viz, that of the latus rectum, to the transverse diameter. So in non-parabolic curves of the next superior order, the solid under the three ordinates, is to the solid under the three abscissas, or the distances to the three vertices, in a certain given ratio. In which ratio if there be taken three lines proportional to the three diameters, each to each; then each of these three lines may be called a *latus rectum*, and each of the corresponding diameters a *transverse diameter*. And, in the common, or Apollonian parabola, which has but one vertex for one diameter, the rectangle of the ordinates is equal to the rectangle of the abscissa and latus rectum: so, in those curves of the second kind, or lines of the third kind, which have

$x = 0$: and in that case $y = \sqrt[3]{(a^2 \times 0)}$, that is, $y = 0$. Therefore the curve passes through A. Let it next be ascertained whether the curve cuts the axis AC in any other point; in order to which, find the value of x when $y = 0$: this will be $\sqrt[3]{a^2x} = 0$, or $x = 0$. Consequently the curve does not cut the axis in any other point than A. Make $x = \text{AP} = \frac{1}{2}a$, and the given equa. will become $y = \sqrt[3]{\frac{1}{2}a^3} = a\sqrt[3]{\frac{1}{2}}$. Therefore draw PM parallel to AB and equal to $a\sqrt[3]{\frac{1}{2}}$, so will M be a point in the curve. Again, make $x = \text{AC} = a$; then the equation will give $y = \sqrt[3]{a^3} = a$. Hence, drawing CN parallel to AB, and equal to AC or a, N will be another point in the curve. And by assuming other values of y, other ordinates, and consequently other points of the curve, may be obtained. Once more, making x infinite, or $x = \infty$, we shall have $y = \sqrt[3]{(a^2 \times \infty)}$; that is, y is infinite when x is so; and therefore the curve passes on to infinity. And further, since when x is taken $= 0$, it is also $y = 0$, and when $x = \infty$, it is also $y = \infty$; the curve will have no asymptotes that are parallel to the co-ordinates.

Let the right line AN be drawn to cut PM (produced if necessary) in S. Then because CN $=$ AC, it will be PS $=$ AP $= \frac{1}{2}a$. But PM $= a\sqrt[3]{\frac{1}{2}} = \frac{1}{2}a\sqrt[3]{4}$, which is manifestly greater than $\frac{1}{2}a$; so that PM is greater than PS, and consequently the curve is concave to the axis AC.

Now, because in the given equation $y^3 = a^2x$ the exponent of x is *odd*, when x is taken negatively or on the other side of A, its sign should be changed, and the reduced equation will then be $y = \sqrt[3]{-a^2x}$. Here it is evident that, when the values of x are taken in the negative way from A towards D, but equal to those already taken the positive way, there will result as many negative values of y, to fall below AD, and each equal to the corresponding values of y, taken above AC. Hence it follows that the branch AM′N′ will be similar and equal to the branch AMN; but contrarily posited.

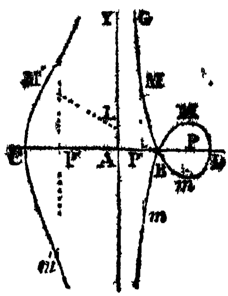
Ex. 2. Let the *lemniscate* be proposed, which is a line of the fourth order, denoted by the equation $a^2y^2 = a^2x^2 - x^4$.



In this equation we have $y = \pm \frac{x}{a} \sqrt{(a^2 - x^2)}$; where, when $x = 0$, $y = 0$, therefore the curve passes through A, the point from which the values of x are measured. When $x = \pm a$, then $y = 0$; therefore the curve passes through B and C, supposing AB and AC each $= \pm a$. If x were assumed greater than a, the value of y would become imaginary; therefore no part of the curve lies beyond B or C. When $x = \frac{1}{2}a$, then

then $y = \frac{1}{2}\sqrt{a^2 - \frac{1}{4}a^2} = \frac{1}{4}a\sqrt{3}$; which is the value of the semiordinate PM when AP $= \frac{1}{2}$AB. And thus, by assuming other values of x, other values of y may be ascertained, and the curve described. It has obviously two equal and similar parts, and a double point at A. A right line may cut this curve in either 2 points, or in 4: even the right line BAC is conceived to cut it in 4 points; because the double point A is that in which two branches of the curve, viz, MAp, and nAQ, are intersected.

Ex. 8. Let there be proposed the *Conchoid* of the ancients, which is a line of the fourth order defined by the equation $(a^2 - x^2) \cdot (x - b)^2 = x^2y^2$, or $y = \pm \frac{x - b}{x}\sqrt{(a^2 - x^2)}$.



Here, if $x = 0$, then y becomes infinite, and therefore the ordinate at A (the origin of the abscissas) is an asymptote to the curve. If AB $= b$, and P be taken between A and B, then shall PM and pm be equal, and lie on different sides of the abscissa AP. If $x = b$, then the two values of y vanish, because $x - b = 0$, and consequently the curve passes through B, having there a double point. If AP be taken greater than AB, then will there be two values of y, as before, having contrary signs; that value which was positive before being now negative, and *vice versa*. But if AD be taken $= a$, and P comes to D, then the two values of y vanish, because in that case $\sqrt{(a^2 - x^2)} = 0$. If AP be taken greater than AD or a, then $a^2 - x^2$ becomes negative, and the value of y impossible: so that the curve does not go beyond D.

Now let x be considered as negative, or as lying on the side of A towards C. Then $y = \pm \frac{x + b}{x}\sqrt{(a^2 - x^2)}$. Here if x vanish, both these values of y become infinite; and consequently the curve has two indefinite arcs on each side the asymptote or directrix AY. If x increase, y manifestly diminishes; and when $x = a$, then y vanishes: that is, if AP $=$ AD, then one branch of the curve passes through C, while the other passes through D. Here also, if x be taken greater than a, y becomes imaginary; so that no part of the curve can be found beyond C.

If $a = b$, the curve will have a cusp in B, the node between B and D vanishing in that case. If a be less than b, then B will become a conjugate point.

In

In the figure, $\text{M}'c m'$ represents what is termed the *superior conchoid*, and GBmDMBm the *inferior conchoid*. The point B is called the *pole* of the conchoid; and the curve may be readily constructed by radial lines from this point, by means of the polar equation $z = \frac{b}{\cos.\,\phi} \pm a$. It will merely be requisite to set off from any assumed point A, the distance $\text{AB} = b$; then to draw through B a right line $m\text{LM}'$ making any angle ϕ with CB, and from L, the point, where this line cuts the directrix AY (drawn perpendicular to CB) set off upon it $\text{LM}' = \text{L}m = a$; so shall M' and m be points in the superior and inferior conchoids respectively.

Ex. 4. Let the principal properties of the curve whose equation is $yx^n = a^{n+1}$, be sought; when n is an odd number, and when n is an even number.

Ex. 5. Describe the line which is defined by the equation $xy + ay + cy = bc + bx$.

Ex. 6. Let the Cardioide, whose equation is $y^4 - 6ay^3 + (2x^2 + 12a^2)y^2 - (6ax^2 + 8a^3)y + (x^2 + 3a^2)x^2 = 0$, be proposed.

Ex. 7. Let the Trident, whose equation is $xy = ax^3 + bx^2 + cx + d$, be proposed.

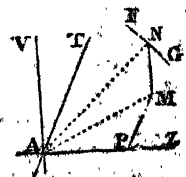
Ex. 8. Ascertain whether the *Cissoid* and the *Witch*, whose equations are found in the preceding problem, have asymptotes.

PROBLEM III.

To determine the Equation to any proposed Curve Surface.

Here the required equation must be deduced from the law or manner of construction of the proposed surface, the reference being to *three* co-ordinates, commonly rectangular ones, the variable quantities being x, y, and z. Of these, two, namely x and y, will be found in one plane, and the third z will always mark the distance from that plane.

Ex. 1. Let the proposed surface be that of a sphere, FNG.



The position of the fixed point A, which is the origin of the co-ordinates AP, PM, MN, being arbitrary; let it be supposed, for the greater convenience, that it is at the centre of the sphere. Let MA, NA, be drawn, of which the latter is manifestly equal to the radius of the sphere, and may be denoted by r. Then, if $\text{AP} = x$, $\text{PM} = y$, $\text{MN} = z$; the right-angled triangle APM will give

AM²

$AM^2 = AP^2 + PM^2 = x^2 + y^2$. In like manner, the right-angled triangle AMN, posited in a plane perpendicular to the former, will give $AN^2 = AM^2 + MN^2$, that is, $r^2 = x^2 + y^2 + z^2$; or $z^2 = r^2 - x^2 - y^2$, the equation to the spherical surface, as required.

Scholium. Curve surfaces, as well as plane curves, are arranged in orders according to the dimensions of the equations, by which they are represented. And, in order to determine the properties of curve surfaces, processes must be employed, similar to those adopted when investigating the properties of plane curves. Thus, in like manner as in the theory of curve lines, the supposition that the ordinate y is equal to 0, gives the point or points where the curve cuts its axis; so, with regard to curve surfaces, the supposition of $z = 0$, will give the equation of the curve made by the intersection of the surface and its base, or the plane of the co-ordinates x, y. Hence, in the equation to the spherical surface, when $z = 0$, we have $x^2 + y^2 = r^2$, which is that of a circle whose radius is equal to that of the sphere. See p. 31.

Ex. 2. Let the curve surface proposed be that produced by a parabola turning about its axis.

Here the abscissas x being reckoned from the vertex or summit of the axis, and on a plane passing through that axis; the two other co-ordinates being, as before, y and z; and the parameter of the generating parabola being p: the equation of the parabolic surface will be found to be $z^2 + y^2 - px = 0$.

Now, in this equation, if z be supposed $= 0$, we shall have $y^2 = px$, which (pa. 31) is the equation to the generating parabola, as it ought to be. If we wished to know what would be the curve resulting from a section parallel to that which coincides with the axis, and at the distance a from it, we must put $z = a$; this would give $y^2 = px - a^2$, which is still an equation to a parabola, but in which the origin of the abscissas is distant from the vertex, before assumed by the quantity $\frac{a^2}{p}$.

Ex. 3. Suppose the curve surface of a right cone were proposed.

Here we may most conveniently refer the equation of the surface to the plane of the circular base of the cone. In this case, the perpendicular distance of any point in the surface from the base, will be to the axis of the cone, as the distance of the foot of that perpendicular from the circumference

 (measured

(measured on a radius), to the radius of the base; that is, if the values of x be estimated from the centre of the base, and r be the radius, z will vary as $r - \surd(x^2 + y^2)$. Consequently, the simplest equation of the conic surface, will be $z \sim r = - \surd(x^2 + y^2)$, or $r^2 - 2rz + z^2 = x^2 + y^2$.

Now, from this, the nature of curves formed by planes cutting the cone in different directions, may readily be inferred. Let it be supposed, first, that the cutting plane is inclined to the base of a right-angled cone in the angle of 45°, and passes through its centre: then will $z = x$, and this value of z substituted for it in the equation of the surface, will give $r^2 - 2rx = y^2$, which is the equation of the projection of the curve on the plane of the cone's base: and this (art. 3 of this chap.) is manifestly an equation to a *parabola*.

Or, taking the thing more generally, let it be supposed that the cutting plane is so situated, that the ratio of x to z shall be that of 1 to m: then will $mx = z$, and $m^2x^2 = z^2$. These substituted for z and z^2 in the equation of the surface, will give, for the equation of the projection of the section on the plane of the base, $r^2 - 2mx + (m^2 - 1)x^2 = y^2$. Now this equation, if m be greater than unity, or if the cutting plane pass between the vertex of the cone and the parabolic section, will be that of an *hyperbola*: and if, on the contrary, the cutting plane pass between the parabola and the base, i. e. if m be less than unity, the term $(m^2 - 1)x^2$ will be negative, when the equation will obviously designate an *ellipse*.

Schol. It might here be demonstrated, in a nearly similar manner, that every surface formed by the rotation of any conic section on one of its axes, being cut by any plane whatever, will always give a conic section. For the equation of such surface will not contain any power of x, y, or z, greater than the second; and therefore the substitution of any values of z in terms of x or of y, will never produce any powers of x or of y exceeding the square. The section therefore must be a line of the second order. See, on this subject, Hutton's Mensuration, part iii, sect. 4.

Ex. 3. Let the equation to the curve surface be $xyz = a^3$.

Then will the curve surface bear the same relation to the *solid* right angle, which the curve line whose equation is $xy = a^2$ bears to the *plane* right angle. That is, the curve surface will be posited between the three rectangular faces bounding such solid right angle, in the same manner as the equilateral hyperbola is posited between its rectangular asymptotes. And in like manner as there may be 4 equal equilateral

teral hyperbolas comprehended between the same rectangular asymptotes, when produced both ways from the angular point; so there may be 6 equal hyperboloids posited within the 6 solid right angles which meet at the same summit, and all placed between the same three asymptotic planes.

SECTION II.

On the Construction of Equations.

PROBLEM I.

To Construct Simple Equations, Geometrically.

HERE the sole art consists in resolving the fractions, to which the unknown quantity is equal, into proportional terms; and then constructing the respective proportions, by means of probs. 8, 9, 10, and 27 Geometry. A few simple examples will render the method obvious.

1. Let $x = \frac{ab}{c}$; then $c : a :: b : x$. Whence x may be found by constructing according to prob. 9 Geometry.

2. Let $x = \frac{abc}{de}$. First construct the proportion $d : a :: b : \frac{ab}{d}$, which 4th term call g; then $x = \frac{gc}{e}$; or $e : c :: g : x$.

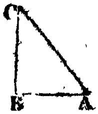
3. Let $x = \frac{a^2 - b^2}{c}$. Then, since $a^2 - b^2 = (a+b) \times (a-b)$; it will merely be necessary to construct the proportion $c : a + b :: a - b : x$.

4. Let $x = \frac{a^2 b - bc^2}{ad}$. Find, as in the first case, $g = \frac{ab}{d} = \frac{a^2 b}{ad}$, and $h = \frac{bc}{d}$, so that $\frac{bc^2}{ad}$ may $= \frac{hc}{a}$. Then find by the first case $i = \frac{hc}{a}$. So shall $x = g - i$, the difference of those lines, found by construction.

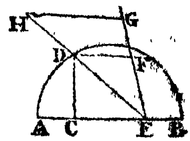
5. Let $x = \frac{a^2 b - bad}{af + bc}$. First find $\frac{af}{b}$, the fourth proportional to b, a and f, which make $= h$. Then $x = \frac{a(a-d)}{h+c}$; or, by construction it will be $h + c : a - d :: a : x$.

6. Let $x = \frac{a^2 + b^2}{c}$. Make the right-angled triangle ABC such

that



that the leg $AB = a$, $BC = b$; then $AC = \surd(AB^2 + BC^2) = \surd(a^2 + b^2)$, by th. 34 Geom. Hence $x = \frac{AC^2}{c}$. Construct therefore the proportion $c : AC :: AC : x$, and the unknown quantity will be found, as required.

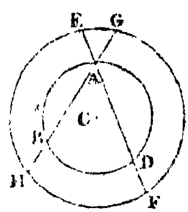


7. Let $x = \frac{a^2 + cd}{h + c}$. First, find CD a mean proportional between $AC = c$, and $CB = d$, that is, find $CD = \surd cd$. Then make $CE = a$, and join DE, which will evidently be $= \surd(a^2 + cd)$. Next on any line EG set off $EF = h + c$, $EG = ED$; and draw GH parallel to FD, to meet DE (produced if need be) in H. So shall EH be $= x$, the third proportional to $h + c$, and $\surd(a^2 + cd)$, as required.

Note. Other methods suitable to different cases which may arise are left to the student's invention. And in all constructions the accuracy of the results, will increase with the size of the diagrams; within convenient limits for operation.

PROBLEM II.

To Find the Roots of Quadratic Equations by Construction.



In most of the methods commonly given for the construction of quadratics, it is required to set off the square root of the last term; an operation which can only be performed accurately when that term is a rational square. We shall here describe a method which, at the same time that it is very simple in practice, has the advantage of showing clearly the relations of the roots, and of dividing the third term into two factors, one of which at least may be a whole number.

In order to this construction, all quadratics may be classed under 4 forms: viz,

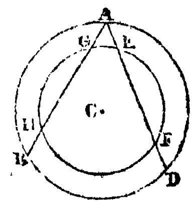
1. $x^2 + ax - bc = 0.$
2. $x^2 - ax - bc = 0.$
3. $x^2 + ax + bc = 0.$
4. $x^2 - ax + bc = 0.$

1. One general mode of construction will include the first two of these forms. Let $x^2 \mp ax - bc = 0$, and b be greater than c. Describe any circle ABD having its diameter not less than the given quantities a and $b - c$, and within this circle inscribe

inscribe two chords, AB $= a$, AD $= b - c$, both from any common assumed point A. Then, produce AD to F so that DF $= c$, and about the centre C of the former circle, with the radius CF, describe another circle, cutting the chords AD, AB, produced, in F, E, G, H: so shall AG be the *affirmative* and AH the *negative root* of the equation $x^2 + ax - bc = 0$; and contrariwise AG will be the *negative* and AH the *affirmative* root of the equation $x^2 - ax - bc = 0$.

For, AF or AD + DF $= b$, and DF or AE $= c$; and, making AG or BH $= x$, we shall have AH $= a + x$: and by the property of the circle EGFH (theor. 61 Geom.) the rectangle EA . AF $=$ GA . AH, or $bc = (a + x)x$, or again by transposition $x^2 + ax - bc = 0$. Also if AH be $= -x$, we shall have AG or BH or AH $-$ AB $= - x - a$: and conseq. GA . AH $= x^2 + ax$, as before. So that, whether AG be $= x$, or AH $= - x$, we shall always have $x^2 + ax - bc = 0$. And by an exactly similar process it may be proved that AG is the negative, and AH the positive root of $x^2 - ax - bc = 0$.

Cor. In quadratics of the form $x^2 + ax - bc = 0$, the positive root is always *less* than the negative root; and in those of the form $x^2 - ax - bc = 0$, the positive root is always *greater* than the negative one.



2. The third and fourth cases also are comprehended under one method of construction, with two concentric circles. Let $x^2 \mp ax + bc = 0$. Here describe any circle ABD, whose diameter is not less than either of the given quantities a and $b + c$; and within that circle inscribe two chords AB $= a$, AD $= b + c$, both from the same point A. Then in AD assume DF $= c$, and about C the centre of the circle ABD, with the radius CF describe a circle, cutting the chords AD, AB, in the points F, E, G, H: so shall AG, AH, be the two *positive* roots of the equation $x^2 - ax + bc = 0$, and the two *negative* roots of the equation $x^2 + ax + bc = 0$. The demonstration of this also is similar to that of the first case.

Cor. 1. If the circle whose radius is CF just touches the chord AB, the quadratic will have two equal roots; which can only happen when $\frac{1}{4}a^2 = bc$.

Cor. 2. If that circle neither cut nor touch the chord AB, the roots of the equation will be imaginary; and this will always happen, in these two forms, when bc is greater than $\frac{1}{4}a^2$

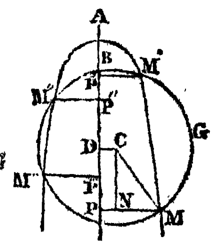
PROBLEM

PROBLEM III.

To Find the Roots of Cubic and Biquadratic Equations, by Construction.

1. In finding the roots of any equation, containing only one unknown quantity, by construction, the contrivance consists chiefly in bringing a new unknown quantity into that equation; so that various equations may be had, each containing the two unknown quantities; and further, such that any two of them contain *together* all the known quantities of the proposed equation. Then from among these equations two of the most simple are selected, and their corresponding loci constructed; the intersection of those loci will give the roots sought.

Thus it will be found that cubics may be constructed by two parabolas, or by a circle and a parabola, or by a circle and an equilateral hyperbola, or by a circle and an ellipse, &c: and biquadratics by a circle and a parabola, or by a circle and an ellipse, or by a circle and an hyperbola, &c. Now, since a parabola of given parameter may be easily constructed by the rule in cor. 2 th. 4 Parabola, we select the circle and the parabola, for the construction of both biquadratic and cubic equations. The general method applicable to both, will be evident from the following description.



2. Let M″AM′M be a parabola whose axis is AP, M″M′GM a circle whose centre is C and radius CM, cutting the parabola in the points M, M′, M″, M‴: from these points draw the ordinates to the axis MP, M′P′, M″P″, M‴P‴; and from C let fall CD perpendicularly to the axis; also draw CN parallel to the axis, meeting PM in N. Let AD $= a$, DC $= b$, CM $= n$, the parameter of the parabola $= p$, AP $= x$, PM $= y$. Then (pa. 31) $px = y^2$: also $\text{CM}^2 = \text{CN}^2 + \text{NM}^2$, or $n^2 = (x \mp a)^2 + (y \mp b)^2$; that is, $x^2 \pm 2ax + a^2 + y^2 \pm 2by + b^2 = n^2$. Substituting in this equation for x, its value $\frac{y^2}{p}$, and arranging the terms according to the dimensions of y, there will arise

$$y^4 \pm (2pa + p^2)y^2 \pm 2bp^2 y + (a^2 + b^2 - n^2)p^2 = 0,$$

a biquadratic equation, whose roots will be expressed by the ordinates PM, P′M′, P″M″, P‴M‴, at the points of intersection of the given parabola and circle.

3. To make this coincide with any proposed biquadratic whose second term is taken away (by cor. theor. 3); assume

$y^4 -$

$y^4 - qy^2 + ry - s = 0$. Assume also $p = 1$; then comparing the terms of the two equations, it will be, $2a - 1 = q$, or $a = \frac{q+1}{2}$, $-2b = r$, or $b = \frac{-r}{2}$; $a^2 + b^2 - n^2 = -s$, or $n^2 = a^2 + b^2 + s$, and consequently $n = \sqrt{(a^2 + b^2 + s)}$. Therefore describe a parabola whose parameter is 1, and in the axis take AD $= \frac{q+1}{2}$: at right angles to it draw DC and $= -\frac{1}{2}r$; from the centre C, with the radius $\sqrt{(a^2 + b^2 + s)}$, describe the circle M″M′GM, cutting the parabola in the points M, M′, M″, M‴; then the ordinates PM, P′M′, P″M″, P‴M‴, will be the roots required.

Note. This method, of making $p = 1$, has the obvious advantage of requiring only one parabola for any number of biquadratics, the necessary variation being made in the radius of the circle.

Cor. 1. When DC represents a negative quantity, the ordinates on the same side of the axis with C represent the negative roots of the equation; and the contrary.

Cor. 2. If the circle *touch* the parabola, two roots of the equation are equal; if it cut it only in two points, or touch it in one, two roots are impossible; and if the circle fall wholly within the parabola, all the roots are impossible.

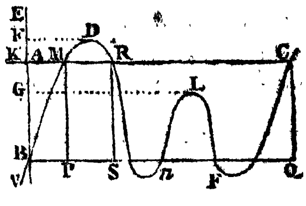
Cor. 3. If $a^2 + b^2 = n^2$, or the circle pass through the point A, the last term of the equation, i. e. $(a^2 + b^2 - n^2)p^2 = 0$; and therefore $y^4 \pm (2pa + p^2)y^2 \pm 2bp^2y = 0$, or $y^3 \pm (2pa + p^2)y \pm 2bp^2 = 0$. This cubic equation may be made to coincide with any proposed cubic, wanting its second term, and the ordinates PM, P″M″, P‴M‴, are its roots.

Thus, if the cubic be expressed generally by $y^3 \pm qy \pm s = 0$. By comparing the terms of this and the preceding equation, we shall have $\pm 2pa + p^2 = \pm q$, and $\pm 2bp^2 = \pm s$, or $\mp a = \frac{1}{2}p \mp \frac{q}{2p}$, and $b = \pm \frac{s}{2p^2}$. So that, to construct a cubic equation, with any *given* parabola, whose half parameter is AB (see the preceding figure): from the point B take, in the axis, (forward if the equation have $-q$, but backward if q be positive) the line BD $= \frac{q}{2p}$; then raise the perpendicular DC $= \frac{s}{2p^2}$, and from C describe a circle passing through the vertex A of the parabola; the ordinates PM, &c, drawn from the points of intersection of the circle and parabola, will be the roots required.

PROBLEM

PROBLEM IV.

To Construct an Equation of any Order by means of a Locus of the same Degree as the Equation proposed, and a Right Line.



As the general method is the same in all equations, let it be one of the 5th degree, as $x^5 - bx^4 + acx^3 - a^2dx^2 + a^3ex - a^4f = 0$. Let the last term a^4f be transposed; and, taking one of the linear divisors, f, of the last term, make it equal to z, for example, and divide the equation by a^4; then will $z = \frac{x^5 - bx^4 + acx^3 - a^2dx^2 + a^3ex}{a^4}$.

On the indefinite line BQ describe the curve of this equation, BMDRLFC, by the method taught in prob. 2, sect. 1, of this chapter, taking the values of x from the fixed point B. The ordinates PM, SR, &c, will be equal to z; and therefore, from the point B draw the right line BA $= f$, parallel to the ordinates PM, SR, and through the point A draw the indefinite right line KC both ways, and parallel to BQ. From the points in which it cuts the curve, let fall the perpendiculars MP, RS, CQ; they will determine the abscissas BP, BS, BQ, which are the roots of the equation proposed. Those from A towards Q are positive, and those lying the contrary way are negative.

If the right line AC touch the curve in any point, the corresponding abscissa x will denote two equal roots; and if it do not meet the curve at all, all the roots will be imaginary.

If the sign of the last term, a^4f, had been positive, then we must have made $z = -f$, and therefore must have taken BA $= -f$, that is, below the point P, or on the negative side.

EXERCISES.

Ex. 1. Let it be proposed to divide a given arc of a circle into three equal parts.

Suppose the radius of the circle to be represented by r, the sine of the given arc by a, the unknown sine of its third part by x, and let the known arc be $3u$, and of course the required arc be u. Then, by equa. VIII, IX, chap. iii, we shall have

$$\sin 3u = \sin(2u + u) = \frac{\sin 2u \cdot \cos u + \cos 2u \cdot \sin u}{r},$$

$$\sin 2u = \sin(u + u) = \frac{2 \sin u \cdot \cos u}{r},$$

$$\cos 2u = \cos(u + u) = \frac{\cos^2 u - \sin^2 u}{r}.$$

Putting,

Putting, in the first of these equations, for sin $3u$ its given value a, and for sin $2u$, cos $2u$, their values given in the two other equations, there will arise

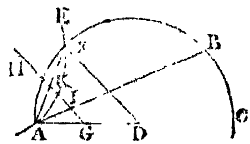
$$a = \frac{3 \sin u \cos^2 u - \sin^3 u}{r}.$$

Then substituting for sin u its value x, and for $\cos^2 u$ its value $r^2 - x^2$, and arranging all the terms according to the powers of x, we shall have

$$x^3 - \tfrac{3}{4}r^2x + \tfrac{1}{4}ar^2 = 0,$$

a cubic equation of the form $x^3 - px + q = 0$, with the condition that $\frac{1}{27}p^3 > \frac{1}{4}q^2$; that is to say, it is a cubic equation falling under the irreducible case, and its three roots are represented by the sines of the three arcs u, $u + 120°$, and $u + 240°$.

Now, this cubic may evidently be constructed by the rule in prob. 3 cor. 3. But the trisection of an arc may also be effected by means of an equilateral hyperbola, in the following manner.



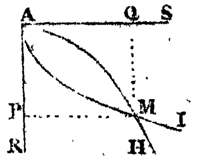
Let the arc to be trisected be AB. In the circle ABC draw the semidiameter AD, and to AD as a diameter, and to the vertex A, draw the equilateral hyperbola AE to which the right line AB (the chord of the arc to be trisected) shall be a tangent in the point A; then the arc AF, included within this hyperbola, is one third of the arc AB.

For, draw the chord of the arc AF, bisect AD at G, so that G will be the centre of the hyperbola, join DF, and draw GH parallel to it, cutting the chords AB, AF, in I and K. Then, the hyperbola being equilateral, or having its transverse and conjugate equal to one another, it follows from Def. 16 Conic Sections, that every diameter is equal to its parameter, and from cor. theor. 2 Hyperbola, that GK . KI = AK^2, or that GK : AK : : AK : KI; therefore the triangles GKA, AKI are similar, and the angle KAI = AGK, which is manifestly = ADF. Now the angle ADF at the centre of the circle being equal to KAI or FAB; and the former angle at the centre being measured by the arc AF, while the latter at the circumference is measured by half FB; it follows that AF = $\frac{1}{2}$FB, or = $\frac{1}{3}$AB, as it ought to be.

Ex. 2. Given the side of a cube, to find the side of another of double capacity.

Let the side of the given cube be a, and that of a double one y, then $2a^3 = y^3$, or, by putting $2a = b$, it will be $a^2b = y^3$: there are therefore to be found two mean proportionals between

tween the side of the cube and twice that side, and the first of those mean proportionals will be the side of the double cube. Now these may be readily found by means of two parabolas; thus:



Let the right lines AR, AS, be joined at right angles; and a parabola AMH be described about the axis AR, with the parameter a; and another parabola AMI about the axis AS, with the parameter b; cutting the former in M. Then AP $= x$, PM $= y$, are the two mean proportionals, of which y is the side of the double cube required.

For, in the parabola AMH the equation is $y^2 = ax$, and in the parabola AMI it is $x^2 = by$. Consequently $a : y :: y : x$, and $y : x :: x : b$. Whence $yx = ab$; or, by substitution, $y\sqrt{by} = ab$, or, by squaring, $y^3b = a^2b^2$; or lastly, $y^3 = a^2b = 2a^3$, as it ought to be.

Note. For other exercises of the construction of equations, take some of the examples at the end of chap. viii.

GENERAL SCHOLIUM.

On the Construction of Geometrical Problems.

Problems in Plane Geometry are solved either by means of the modern or algebraical analysis, or of the ancient or geometrical analysis. Of the former, some specimens are given in the Application of Algebra to Geometry, in the first volume of this Course. Of the latter, we here present a few examples, premising a brief account of this kind of analysis.

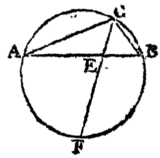
Geometrical analysis is the way by which we proceed from the thing demanded, granted for the moment, till we have connected it by a series of consequences with something anteriorly known, or placed it among the number of principles known to be true.

Analysis may be distinguished into two kinds. In the one, which is named by Pappus contemplative, it is proposed to ascertain the truth or the falsehood of a proposition advanced; the other is referred to the solution of problems, or to the investigation of unknown truths. In the first we assume as true, or as previously existing, the subject of the proposition advanced, and proceed by the consequences of the hypothesis to something known; and if the result be thus found true, the proposition advanced is likewise true. The direct demonstration is afterwards formed, by taking up again, in an inverted order, the several parts of the analysis. If the consequence at which we arrive in the last place is found false, we

we thence conclude that the proposition analysed is also false. When a *problem* is under consideration, we first suppose it resolved, and then pursue the consequences thence derived till we come to something known. If the ultimate result thus obtained be comprised in what the geometers call data, the question proposed may be resolved: the demonstration (or rather the construction), is also constituted by taking the parts of the analysis in an inverted order. The impossibility of the last result of the analysis, will prove evidently, in this case as well as in the former, that of the thing required.

In illustration of these remarks take the following examples.

Ex. 1. It is required to draw, in a given segment of a circle, from the extremes of the base A and B, two lines AC, BC, meeting at a point C in the circumference, such that they shall have to each other a given ratio, viz, that of M to N.

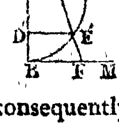


Analysis. Suppose that the thing is effected, that is to say, that AC : CB : : M : N, and let the base AB of the segment be cut in the same ratio in the point E. Then EC, being drawn, will bisect the angle ACB (by th. 83 Geom.); consequently, if the circle be completed, and CE be produced to meet it in F, the remaining circumference will also be bisected in F, or have FA = FB, because those arcs are the double measures of equal angles: therefore the point F, as well as E, being given, the point C is also given.

Construction. Let the given base of the segment AB be cut in the point E in the assigned ratio of M to N, and complete the circle; bisect the remaining circumference in F; join FE, and produce it till it meet the circumference in C: then drawing CA, CB, the thing is done.

Demonstration. Since the arc FA = the arc FB, the angle ACF = angle BCF, by theor. 49 Geom.; therefore AC : CB : : AE : EB, by th. 83. But AE : EB : : M : N, by construction; therefore AC : CB : : M : N. Q. E. D.

Ex. 2. From a given circle to cut off an arc such, that the sum of m times the sine, and n times the versed sine, may be equal to a given line.



Anal. Suppose it done, and that AEE′B is the given circle, BE′E the required arc, ED its sine, BD its versed sine; in DA (produced if necessary) take BP an nth part of the given sum; join PE, and produce it to meet BF ⊥ to AB, or ∥ to ED, in the point F. Then, since $m \cdot \text{ED} + n \cdot \text{BD} = n \cdot \text{BP} = n \cdot \text{PD} + n \cdot \text{BD}$;

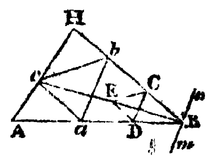
consequently

consequently $m \cdot ED = n \cdot PD$; hence $PD : ED :: m : n$. But $PD : ED ::$ (by sim. tria.) $PB : BF$; therefore $PB : BF :: m : n$ Now PB is given, therefore BF is given in magnitude, and, being at right angles to PB, is also given in position; therefore the point F is given, and consequently PF given in position; and therefore the point E, its intersection with the circumference of the circle AEB'B, or the arc BE is given. Hence the following

Const. From B, the extremity of any diameter AB of the given circle, draw BM at right angles to AB; in AB (produced if necessary) take BP an nth part of the given sum; and on BM take BF so that $BF : BP :: n : m$. Join PF, meeting the circumference of the circle in E and E', and BE or BE' is the arc required.

Demon. From the points E and E' draw ED and E'D' at right angles to AB. Then, since $BF : BP :: n : m$, and (by sim. tri.) $BF : BP :: DE : DP$; therefore $DE : DP :: n : m$. Hence $m \cdot DE = n \cdot DP$; add to each $n \cdot BD$, then will $m \cdot DE + n \cdot BD = n \cdot BD + n \cdot DP = n \cdot PB$, or the given sum.

Ex. 3. In a given triangle ABH, to inscribe another triangle abc, similar to a given one, having one of its sides parallel to a line mBn given by position, and the angular points a, b, c, situate in the sides AB, BH, AH, of the triangle ABH respectively.



Analysis. Suppose the thing done, and that abc is inscribed as required. Through any point C in BH draw CD parallel to mBn or to ab, and cutting AB in D; draw CE parallel to bc, and DE to ac, intersecting each other in E. The triangles DEC, acb, are similar, and $DC : ab :: CE : bc$; also BDC, Bab, are similar, and $DC : ab :: BC : Bb$. Therefore $BC : CE :: Bb \cdot bc$; and they are about equal angles, consequently B, E, c, are in a right line.

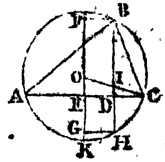
Construc. From any point C in BH, draw CD parallel to nm; on CD constitute a triangle CDE similar to the given one; and through its angle E draw BE, which produce till it cuts AH in c: through c draw ca parallel to ED and cb parallel to EC; join ab, then abc is the triangle required, having its side ab parallel to mn, and being similar to the given triangle.

Demon. For, because of the parallel lines ac, DE, and cb, EC, the quadrilaterals BDEC and Bacb, are similar; and therefore the proportional lines DC, ab, cutting off equal angles

BDC,

BDC, *Bab*; BCD, *Bba*; must make the angles EDC, ECD, respectively equal to the angles *cab*, *cba*; while *ab* is parallel to DC, which is parallel to *m*B*n*, by construction.

Ex. 4. Given, in a plane triangle, the vertical angle, the perpendicular, and the rectangle of the segments of the base, made by that perpendicular; to construct the triangle.



Anal. Suppose ABC the triangle required, BD the given perpendicular to the base AC, produce it to meet the periphery of the circumscribing circle ABCH, whose centre is O, in H; then, by th. 61 Geom. the rectangle BD . DH = AD . DC, the given rectangle: hence, since BD is given, DH and BH are given; therefore BI = HI is given; as also ID = OE: and the angle EOC is = ABC the given one, because EOC is measured by the arc KC, and ABC by half the arc AKC or by KC. Consequently EC and AC = 2EC are given. Whence this

Construction. Find DH such, that DB . DH = the given rectangle, or find $\text{DH} = \frac{\text{AD . DC}}{\text{BD}}$; then on any right line GF take FE = the given perpendicular, and EG = DH; bisect FG in O, and make EOC = the given vertical angle; then will OC cut EC, drawn perpendicular to OE, in C. With centre O and radius OC, describe a circle, cutting CE produced in A: through F parallel to AC draw FB, to cut the circle in B; join AB, CB, and ABC is the triangle required.

Remark. In a similar manner we may proceed, when it is required to divide a given angle into two parts, the rectangle of whose tangents may be of a given magnitude. See prob. 40, Simpson's Select Exercises.

Note. For other exercises, the student may construct all the problems except the 24th, in the Application of Algebra to Geometry, at the end of vol. 1. And that he may be the better able to trace the relative advantages of the ancient and the modern analysis, it will be adviseable that he solve those problems both geometrically and algebraically.

CHAPTER X.

OF FLUXIONS AND FLUENTS.

Art. 1. In the 2d volume of this Course has been given a compendious and easy treatise on Fluxions and Fluents; and what follows is a further and more general extension of the same subject, chiefly on the transformation and on the inverse method of fluxions; as the rules for the direct method, given in that volume, will be found quite sufficient for finding the fluxions of the ordinary forms of quantities. From art. 32, to art. 48, of that volume, have been given a collection of the most common and obvious rules for finding the fluents of given fluxions; and which require no further proof or consideration, as they are self-evident, being simply the reverse of the preceding rules for finding fluxions. But, in art. 42 &c, is given also a compendious table of various other forms of fluxions and fluents, the truth of which it may be proper here in the first place to prove.

2. As to most of those forms indeed, they will be easily proved, by only taking the fluxions of the forms of fluents, in the last column, by means of the rules before given in art. 30 of the direct method; by which they will be found to produce the corresponding fluxions in the 2d column of the table. Thus, the 1st and 2d forms of fluents will be proved by the 1st of the said rules for fluxions: the 3d and 4th forms of fluents by the 4th rule for fluxions: the 5th and 6th forms, by the 3d rule of fluxions: the 7th, 8th, 9th, 10th, 12th, 14th forms, by the 6th rule of fluxions: the 17th form, by the 7th rule of fluxions: the 18th form, by the 8th rule of fluxions. So that there remains only to prove the 11th, 13th, 15th, and 16th forms.

3. Now, as to the 16th form, that is proved by the 2d example in art. 63, where it appears that $\dot{x}\sqrt{(dx-x^2)}$ is the fluxion of the circular segment, whose diameter is d, and versed sine x. And the remaining three forms, viz, the 11th, 13th, and 15th; will be proved by means of the rectifications of circular arcs, in art. 61.

4. Thus, for the 11th form, it appears by that art. that the fluxion of the circular arc z, whose radius is r and tangent t, is $\dot{z} = \frac{r^2\dot{t}}{r^2+t^2}$. Now put $t = x^{\frac{1}{2}n}$, or $t^2 = x^n$, and $a = r^2$: then is $\dot{t} = \frac{1}{2}nx^{\frac{1}{2}n-1}\dot{x}$, and $r^2 + t^2 = a + x^n$, and $\dot{z} = \frac{r^2\dot{t}}{r^2+t^2}$

$= \frac{1}{2}anx$

$= \frac{\frac{1}{2}anx^{\frac{1}{2}n-1}\dot{x}}{a+x^n}$; hence $\frac{x^{\frac{1}{2}n-1}\dot{x}}{a+x^n} = \frac{\dot{z}}{\frac{1}{2}an} = \frac{2}{an}\dot{z}$, and the fluent is $\frac{2z}{an} = \frac{2}{na} \times$ arc to radius $\sqrt{a}$ and tang. $x^{\frac{1}{2}n}$, or $= \frac{2}{n\sqrt{a}} \times$ arc to radius 1 and tang. $\sqrt{\frac{x^n}{a}}$, which is the first form of the fluent in n°. XI.

5. And, for the latter form of the fluent in the same n°; because the coefficient of the former of these, viz, $\frac{2}{n\sqrt{a}}$ is double of $\frac{1}{n\sqrt{a}}$ the coefficient of the latter, therefore the arc in the latter case, must be double the arc in the former. But the cosine of double an arc, to radius 1 and tangent t, is $\frac{1-t^2}{1+t^2}$; and because $t^2 = \frac{x^n}{a}$ by the former case, this substituted for t^2 in the cosine $\frac{1-t^2}{1+t^2}$, it becomes $\frac{a-x^n}{a+x^n}$, the cosine as in the latter case of the 11th form.

6. Again, for the first case of the fluent in the 13th form. By art. 61 vol. 2, the fluxion of the circular arc z, to radius r and sine y, is $\dot{z} = \frac{r\dot{y}}{\sqrt{(r^2-y^2)}}$ or $= \frac{\dot{y}}{\sqrt{(1-y^2)}}$ to the radius 1. Now put $y = \sqrt{\frac{x^n}{a}}$, or $y^2 = \frac{x^n}{a}$; hence $\sqrt{(1-y^2)} = \sqrt{(1-\frac{x^n}{a})} = \sqrt{\frac{1}{a}} \times \sqrt{(a-x^n)}$, and $\dot{y} = \sqrt{\frac{1}{a}} \times \frac{1}{2}nx^{\frac{1}{2}n-1}\dot{x}$; then these two being substituted in the value of $\dot{z}$, give $\dot{z}$ or $\frac{\dot{y}}{\sqrt{(1-y^2)}} = \frac{n}{2} \times \frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{(a-x^n)}}$; consequently the given fluxion $\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{(a-x^n)}}$ is $= \frac{2}{n}\dot{z}$, and therefore its fluent is $\frac{2}{n}z$, that is $\frac{2}{n} \times$ arc to sine $\sqrt{\frac{x^n}{a}}$, as in the table of forms, for the first case of form XIII.

7. And, as the coefficient $\frac{1}{n}$, in the latter case of the said form, is the half of $\frac{2}{n}$ the coefficient in the former case, therefore the arc in the latter case must be double of the arc in the former. But, by trigonometry, the versed sine of double an arc, to sine y and radius 1, is $2y^2$, and, by the former case, $2y^2 = \frac{2x^n}{a}$; therefore $\frac{1}{n} \times$ arc to the versed sine $\frac{2x^n}{a}$ is the fluent, as in the 2d case of form XIII.

8. Again

8. Again, for the first case of fluent in the 15th form. By art. 61 vol. 2, the fluxion of the circular arc z, to radius r and secant s, is $\dot{z} = \frac{r^2\dot{s}}{s\sqrt{(s^2-r^2)}}$, or $= \frac{\dot{s}}{s\sqrt{(s^2-1)}}$ to radius 1. Now, put $s = \sqrt{\frac{x^n}{a}} = \frac{x^{\frac{1}{2}n}}{\sqrt{a}}$, or $s^2 = \frac{x^n}{a}$; hence $s\sqrt{(s^2-1)} = \frac{x^{\frac{1}{2}n}}{\sqrt{a}}\sqrt{(\frac{x^n}{a}-1)} = \frac{x^{\frac{1}{2}n}}{a}\sqrt{(x^n-a)}$, and $\dot{s} = \sqrt{\frac{1}{a}} \times \frac{1}{2}nx^{\frac{1}{2}n-1}\dot{x}$; then these two being substituted in the value of $\dot{z}$, give $\dot{z}$ or $\frac{\dot{s}}{s\sqrt{(s^2-1)}} = \frac{n\sqrt{a}}{2} \times \frac{x^{-1}\dot{x}}{\sqrt{(x^n-a)}}$; consequently the given fluxion $\frac{x^{-1}\dot{x}}{\sqrt{(x^n-a)}} = \frac{2}{n\sqrt{a}}\dot{z}$, and theref. its fluent is $\frac{2}{n\sqrt{a}}z$, that is $\frac{2}{n\sqrt{a}}$ $\times$ arc to secant $\sqrt{\frac{x^n}{a}}$, as in the table of forms, for the first case of form XV.

9. And, as the coefficient $\frac{1}{n\sqrt{a}}$, in the latter case of the said form, is the half of $\frac{2}{n\sqrt{a}}$, the coefficient of the former case, therefore the arc in the latter case must be double the arc in the former. But, by trigonometry, the cosine of the double arc, to secant s and radius 1, is $\frac{2}{s^2} - 1$; and, by the former case, $\frac{2}{s^2} - 1 = \frac{2a}{x^n} - 1 = \frac{2a-x^n}{x^n}$; therefore $\frac{1}{n\sqrt{a}} \times$ arc to cosine $\frac{2a-x^n}{x^n}$ is the fluent, as in the 2d case of form XV.

Or, the same fluent will be $\frac{2}{n\sqrt{a}} \times$ arc to cosine $\sqrt{\frac{a}{x^n}}$, because the cosine of an arc, is the reciprocal of its secant.

10. It has been just above remarked, that several of the tabular forms of fluents are easily shown to be true, by taking the fluxions of those forms, and finding they come out the same as the given fluxions. But they may also be determined in a more direct manner, by the transformation of the given fluxions to another form. Thus, omitting the first form, as too evident to need any explanation, the 2d form is $\dot{z} = (a + x^n)^{m-1}x^{n-1}\dot{x}$, where the exponent $(n-1)$ of the unknown quantity without the vinculum, is 1 less than (n) that under the same. Here, putting $y =$ the compound quantity $a + x^n$: then is $\dot{y} = nx^{n-1}\dot{x}$, and $\dot{z} = \frac{y^{m-1}\dot{y}}{n}$; hence, by art. 36, $z = \frac{y^m}{mn} = \frac{(a+x^n)^m}{mn}$, as in the table.

11. By

11. By the above example it appears, that such form of fluxion admits of a fluent in finite terms, when the index $(n - 1)$ of the variable quantity (x) without the vinculum, is less by 1 than n, the index of the same quantity under the vinculum. But it will also be found, by a like process, that the same thing takes place in such forms as $(a + x^n)^m x^{cn-1}\dot{x}$, where the exponent $(cn - 1)$ without the vinculum, is 1 less than any multiple (c) of that (n) under the vinculum. And further, that the fluent, in each case, will consist of as many terms as are denoted by the integer number c; viz, of one term when $c = 1$, of two terms when $c = 2$, of three terms when $c = 3$, and so on.

12. Thus, in the general form, $\dot{z} = (a + x^n)^m x^{cn-1}\dot{x}$, putting, as before, $a + x^n = y$; then is $x^n = y - a$, and its fluxion $nx^{n-1}\dot{x} = \dot{y}$, or $x^{n-1}\dot{x} = \frac{\dot{y}}{n}$, and $x^{cn-1}\dot{x}$ or $x^{cn-n}x^{n-1}\dot{x} = \frac{1}{n}(y - a)^{c-1}\dot{y}$; also $(a + x^n)^m = y^m$: these values being now substituted in the general form proposed, give $\dot{z} = \frac{1}{n}(y - a)^{c-1}y^m\dot{y}$. Now, if the compound quantity $(y - a)^{c-1}$ be expanded by the binomial theorem, and each term multiplied by $y^m\dot{y}$, that fluxion becomes

$$\dot{z} = \frac{1}{n}\left(y^{m+c-1}\dot{y} - \frac{c-1}{1}ay^{m+c-2}\dot{y} + \frac{c-1}{1}\cdot\frac{c-2}{2}a^2y^{m+c-3}\dot{y} - \right.$$

&c); then the fluent of every term, being taken by art. 36, it is

$$z = \frac{1}{n}\left(\frac{y^{m+c}}{m+c} - \frac{c-1}{1}\cdot\frac{ay^{m+c-1}}{m+c-1} + \frac{c-1}{1}\cdot\frac{c-2}{2}\cdot\frac{a^2y^{m+c-2}}{m+c-2} - \&c\right),$$

$$= \frac{y^d}{n}\left(\frac{1}{d} - \frac{c-1}{d-1}\cdot\frac{a}{y} + \frac{c-1\,.\,c-2}{d-2}\cdot\frac{a^2}{2y^2} - \frac{c-1\,.\,c-2\,.\,c-3}{d-3}\cdot\frac{a^3}{2\,.\,3y^3}\right.$$

&c), putting $d = m + c$, for the general form of the fluent; where, c being a whole number, the multipliers $c - 1$, $c - 2$, $c - 3$, &c, will become equal to nothing, after the first c terms, and therefore the series will then terminate, and exhibit the fluent in that number of terms; viz, there will be only the first term when $c = 1$, but the first two terms when $c = 2$, and the first three terms when $c = 3$, and so on.—Except however the cases in which m is some negative number equal to or less than c; in which cases the divisors, $m + c$, $m + c - 1$, $m + c - 2$, &c, becoming equal to nothing, before the multipliers $c - 1$, $c - 2$, &c, the corresponding terms of the series, being divided by 0, will be infinite: and then the fluent is said to fail, as in such case nothing can be determined from it.

13. Besides this form of the fluent, there are other methods of proceeding, by which other forms of fluents are

derived, of the given fluxion $\dot{z} = (a + x^n)^m x^{cn-1}\dot{x}$, which are of use when the foregoing form fails, or runs into an infinite series; some results of which are given both by Mr. Simpson and Mr. Landen. The two following processes are after the manner of the former author.

14. The given fluxion being $(a + x^n)^m x^{cn-1}\dot{x}$; its fluent may be assumed equal to $(a + x^n)^{m+1}$ multiplied by a general series, in terms of the powers of x combined with assumed unknown coefficients, which series may be either ascending or descending, that is, having the indices either increasing or decreasing;

viz, $(a + x^n)^{m+1} \times (\text{A}x^r + \text{B}x^{r-s} + \text{C}x^{r-2s} + \text{D}x^{r-3s} + \text{\&c}),$

or $(a + x^n)^{m+1} \times (\text{A}x^r + \text{B}x^{r+s} + \text{C}x^{r+2s} + \text{D}x^{r+3s} + \text{\&c}).$

And first, for the former of these, take its fluxion in the usual way, which put equal to the given fluxion $(a+x^n)^m x^{cn-1}\dot{x}$, then divide the whole equation by the factors that may be common to all the terms; after which, by comparing the like indices and the coefficients of the like terms, the values of the assumed indices and coefficients will be determined, and consequently the whole fluent. Thus, the former assumed series in fluxions is,

$n(m + 1)x^{n-1}\dot{x}(a + x^n)^m \times (\text{A}x^r + \text{B}x^{r-s} + \text{C}x^{r-2s} \text{ \&c}) +$
$(a+x^n)^{m+1}\dot{x} \times (r\text{A}x^{r-1} + (r-s)\text{B}x^{r-s-1} + (r-2s)\text{C}x^{r-2s-1}$
&c); this being put equal to the given fluxion $(a+x^n)^m x^{cn-1}\dot{x}$, and the whole equation divided by $(a+x^n)^m x^{-1}\dot{x}$, there results

$$\left.\begin{array}{l} n(m+1)x^n \times (\text{A}x^r + \text{B}x^{r-s} + \text{C}x^{r-2s} + \text{D}x^{r-3s} + \text{\&c}) \\ +(a+x^n)\times(r\text{A}x^r + (r-s)\text{B}x^{r-s} + (r-2s)\text{C}x^{r-2s} \text{ \&c}) \end{array}\right\} = x^{cn}.$$

Hence, by actually multiplying, and collecting the coefficients of the like powers of x, there results

$$\left.\begin{array}{l} \left.\begin{array}{l} n(m+1) \\ \quad +r \end{array}\right\} \text{A}x^{r+n} + \left.\begin{array}{l} n(m+1) \\ +r-s \end{array}\right\} \text{B}x^{r+n-s} + \left.\begin{array}{l} n(m+1) \\ +r-2s \end{array}\right\} \text{C}x^{r+n-2s} \text{ \&c} \\ \quad -x^{cn} \ldots + \ldots\ldots ra\text{A}x^r \ \ldots + (r-s)a\text{B}x^{r-s} \text{ \&c.} \end{array}\right\} = 0.$$

Here, by comparing the greatest indices of x, in the first and second terms, it gives $r + n = cn$, and $r + n - s = r$; which give $r = (c - 1)n$, and $n = s$. Then these values being substituted in the last series, it becomes

$$\left.\begin{array}{l} (c+m)n\text{A}x^{cn} + (c+m-1)n\text{B}x^{cn-n} + (c+m-2)n\text{C}x^{cn-2n} \text{ \&c} \\ \quad -x^{cn} + (c-1)na\text{A}x^{cn-n} + (c-2)na\text{B}x^{cn-2n} \text{ \&c} \end{array}\right\} = 0$$

Now, comparing the coefficients of the like terms, and putting $c + m = d$, there result these equalities:

$\text{A} = \frac{1}{dn}$; $\text{B} = -\frac{c-1 \,.\, a\text{A}}{d-1} = -\frac{c-1 \,.\, a}{d-1 \,.\, dn}$; $\text{C} = -\frac{c-2 \,.\, a\text{B}}{d-2} = +\frac{c-1 \,.\, c-2 \,.\, a^2}{d-1 \,.\, d-2 \,.\, dn}$;

&c; which values of A, B, C, &c, with those of r and s, being now substituted in the first assumed fluent, it becomes

$(a + x^n)$

$\frac{(a+x^n)^{m+1}x^{cn-n}}{cn} \times (\frac{1}{1} - \frac{c-1 . a}{d-1 . x^n} + \frac{c-1 . c-2 . a^2}{d-1 . d-2 . x^{2n}} - \frac{c-1 . c-2 . c-3 . a^3}{d-1 . d-2 . d-3 . x^{3n}} +$ &c, the true fluent of $(a+x^n)^m x^{cn-1}\dot{x}$, exactly agreeing with the first value of the 19th form in the table of fluents in my Dictionary. Which fluent therefore, when c is a whole positive number, will always terminate in that number of terms; subject to the same exception as in the former case. Thus, if $c = 2$, or the given fluxion be $(a+x^n)^m x^{2n-1}\dot{x}$; then, $c + m$ or d being $= m + 2$, the fluent becomes

$$\frac{(a+x^n)^{m+1}x^n}{(m+2)n} \times (1 - \frac{ax^{-n}}{m+1}) = \frac{(a+x^n)^{m+1}}{n} \times \frac{(m+1)x^n - a}{m+1 . m+2}.$$

And if $c = 3$, or the given fluxion be $(a+x^n)^m x^{3n-1}\dot{x}$; then $m + c$ or d being $= m + 3$, the fluent becomes

$$\frac{(a+x^n)^{m+1}x^{2n}}{(m+3)n} \times (1 - \frac{2ax^{-n}}{m+2} + \frac{2a^2x^{-2n}}{m+2 . m+1}) = \frac{(a+x^n)^{m+1}}{n} \times (\frac{x^{2n}}{m+3} - \frac{2ax^n}{m+3 . m+2} + \frac{2a^2}{m+3 . m+2 . m+1}).$$

And so on, when c is other whole numbers: but, when c denotes either a fraction or a negative number, the series will then be an infinite one, as none of the multipliers $c-1$, $c-2$, $c-3$, can then be equal to nothing.

15. Again, for the latter or ascending form, $(a+x^n)^{m+1} \times (Ax^r + Bx^{r+s} + Cx^{r+2s} + Dx^{r+3s} +$ &c$)$, by making its fluxion equal to the proposed one, and dividing, &c, as before, equating the two least indices, &c, the fluent will be obtained in a different form, which will be useful in many cases, when the foregoing one fails, or runs into an infinite series. Thus, if $r+s$, $r+2s$, &c, be written instead of $r-s$, $r-2s$, &c, respectively, in the general equation in the last case, and taking the first term of the 2d line into the first line, there results

$$\left.\begin{matrix} -x^{cn} + n(m+1) \\ +r \end{matrix}\right\} Ax^{r+n} \begin{matrix} +n(m+1) \\ +r+s \end{matrix}\Big\} Bx^{r+n+s} \text{ \&c} \\ + raAx^r + (r+s)aBx^{r+s} + (r+2s)aCx^{r+2s} \text{ \&c} \Bigg\} = 0.$$

Here, comparing the two least pairs of exponents, and the coefficients, we have $r = cn$, and $s = n$; then $A = \frac{1}{ra} = \frac{1}{cna}$; $B = -\frac{r+n(m+1)}{a(r+s)}$; $A = -\frac{c+m+1}{c+1} . \frac{A}{a} = -\frac{c+m+1}{(c+1)cna^2}$; $C = -\frac{c+m+2}{(c+2)a}B = +\frac{c+m+1 . c+m+2}{c . c+1 . c+2 . na^3}$ &c. Therefore, denoting $c+m$ by d, as before, the fluent of the same fluxion $(a+x^n)^m x^{cn-1}\dot{x}$, will also be truly expressed by

$$\frac{(a+x^n)^{m+1}x^{cn}}{cna} \times (\frac{1}{1} - \frac{d+1 . x^n}{c+1 . a} + \frac{d+1 . d+2 . x^{2n}}{c+1 . c+2 . a^2} - \text{\&c});$$

agreeing with the 2d value of the fluent of the 19th form in

 my

my Dictionary. Which series will terminate when d or $c+m$ is a negative integer; except when c is also a negative integer less than d; for then the fluent fails, or will be infinite, the divisor in that case first becoming equal to nothing.

To show now the use of the foregoing series, in some example of finding fluents, take first,

16. *Example* 1. To find the fluent of

$\frac{6x\dot{x}}{\sqrt{(a+x)}}$ or $6x\dot{x}\,(a+x)^{-\frac{1}{2}}$.

This example being compared with the general form $x^{cn-1}\dot{x}(a+x^n)^m$, in the several corresponding parts of the first series, gives these following equalities: viz, $a=a$, $n=1$, $cn-1=1$, or $c-1=1$, or $c=2$; $m=-\frac{1}{2}$; $y=a+x$, $d=m+c=2-\frac{1}{2}=\frac{3}{2}$, $\frac{1}{n}y^d=(a+x)^{\frac{3}{2}}$, $\frac{1}{d}=\frac{2}{3}$, $\frac{c-1}{d-1}$. $\frac{a}{y}=\frac{2a}{a+x}$; here the series ends, as all the terms after this become equal to nothing, because the following terms contain the factor $c-2=0$. These values then being substituted in $\frac{y^d}{n}(\frac{1}{d}-\frac{c-1}{d-1}\cdot\frac{a}{y})$, it becomes $(a+x)^{\frac{3}{2}}\times$ $(\frac{2}{3}-\frac{2a}{a+x})=(\frac{2a+2x}{3}-2a)\times(a+x)^{\frac{1}{2}}=\frac{2x-4a}{3}\sqrt{(a+x)}$; which multiplied by 6, the given coefficient in the proposed example, there results $(4x-8a)\,.\,\sqrt{(a+x)}$, for the fluent required.

17. *Exam.* 2. To find the fluent of

$$\frac{3\dot{x}\sqrt{(a^2+x^2)}}{x^6}=(a^2+x^2)^{\frac{1}{2}}\times 3x^{-6}\dot{x}.$$

The several parts of this quantity being compared with the corresponding ones of the general form, give $a=a^2$, $n=2$, $m=\frac{1}{2}$, $cn-1$ or $2c-1=-6$, whence $c=\frac{1-6}{2}=-\frac{5}{2}$, and $d=m+c=\frac{1}{2}-\frac{5}{2}=-\frac{4}{2}=-2$, which being a negative integer, the fluent will be obtained by the 3d or last form of series; which, on substituting these values of the letters, gives $\frac{3(a^2+x^2)^{\frac{3}{2}}x^{-5}}{-5a^2}\times(\frac{1}{1}-\frac{-1\,.\,x^2}{-\frac{3}{2}a^2})=\frac{3(a^2+x^2)^{\frac{3}{2}}}{-5a^2x^5}\times(1-\frac{2x^2}{3a^2})=$ $\frac{(a^2+x^2)^{\frac{3}{2}}}{x^5}\times\frac{2x^2-3a^2}{5a^4}$ for the required fluent of the proposed fluxion.

18. *Exam.* 3. Let the fluxion proposed be

$$\frac{5x^{3n-1}\dot{x}}{\sqrt{(b+x^n)}}=5(b+x^n)^{-\frac{1}{2}}x^{3n-1}\dot{x}.$$

Here, by proceeding as before, we have $a=b$, $n=n$, $m=-\frac{1}{2}$, $c=3$, and $d=c+m=\frac{5}{2}$; where c being a positive integer, this case belongs to the 2d series; into which

which therefore the above values being substituted, it becomes $\frac{3(b+x^n)^{\frac{3}{2}}x^{2n}}{\frac{3}{2}n} \times (\frac{1}{1} - \frac{2b}{\frac{3}{2}x^n} + \frac{2\cdot 1\cdot b^2}{\frac{3}{2}\cdot\frac{1}{2}x^{2n}}) = 2\sqrt{(b+x^n)} \times \frac{3x^{2n}-4bx^n+8b^2}{3n}$.

19. *Exam.* 4. Let the proposed fluxion be $5(\frac{1}{3}-z^2)^{\frac{1}{2}}z^{-8}\dot{z}$.

Here, proceeding as above, we have $a=\frac{1}{3}$, $n=2$, $m=\frac{1}{2}$, $cn-1$ or $2c-1=-8$, and $c=-\frac{7}{2}$, $x=-z$, $d=c+m=-3$, which being a negative integer, the case belongs to the 3d or last series; which therefore, by substituting these values, becomes $\frac{5(\frac{1}{3}-z^2)^{\frac{3}{2}}}{-7\cdot\frac{1}{3}z^7} \times (\frac{1}{1} + \frac{-2z^2}{-\frac{5}{2}\cdot\frac{1}{3}} + \frac{-2\cdot -1\cdot z^4}{-\frac{5}{2}\cdot -\frac{3}{2}\cdot\frac{1}{9}}) = \frac{15(\frac{1}{3}-z^2)^{\frac{3}{2}}}{-7z^7} \times (1 + \frac{12z^2}{5} + \frac{24z^4}{5}) = \frac{-3(\frac{1}{3}-z^2)^{\frac{3}{2}}}{7z^7} \times (5 + 12z^2 + 24z^4)$, the true fluent of the proposed fluxion. And thus may many other similar fluents be exhibited in finite terms, as in these following examples for practice.

Ex. 5. To find the fluent of $-3x^3\dot{x}\sqrt{(a^2-x^2)}$.

Ex. 6. To find the fluent of $-6x^5\dot{x}\,.\,(a^2-x^2)^{-\frac{5}{2}}$.

Ex. 7. To find the flu. of $\frac{\dot{x}\sqrt{(a-x^n)}}{x^{\frac{7}{2}n-1}}$ or $(a-x^n)^{\frac{1}{2}}x^{-\frac{7}{2}n+1}\dot{x}$.

20. The case mentioned in art. 37, vol. 2, viz, of compound quantities under the vinculum, the fluxion of which is in a given ratio to the fluxion without the vinculum, with only one variable letter, will equally apply when the compound quantities consist of several variables. Thus,

Example 1. The given fluxion being $(4x\dot{x}+8y\dot{y}) \times \sqrt{(x^2+2y^2)}$, or $(4x\dot{x}+8y\dot{y}) \times (x^2+2y^2)^{\frac{1}{2}}$, the root being x^2+2y^2, the fluxion of which is $2x\dot{x}+4y\dot{y}$. Dividing the former fluxional part by this fluxion, gives the quotient 2: next, the exponent $\frac{1}{2}$ increased by 1, gives $\frac{3}{2}$: lastly, dividing by this $\frac{3}{2}$, there then results $\frac{4}{3}(x^2+2y^2)^{\frac{3}{2}}$, for the required fluent of the proposed fluxion.

Exam. 2. In like manner, the fluent of

$$(x^2+y^4+z^6)^{\frac{1}{3}} \times (6x\dot{x}+12y^3\dot{y}+18z^5\dot{z}) \text{ is}$$

$$\frac{(x^2+y^4+z^6)^{\frac{1}{3}+1} \times (6x\dot{x}+12y^3\dot{y}+18z^5\dot{z})}{(2x\dot{x}+4y^3\dot{y}+6z^5\dot{z}) \times \frac{4}{3}} = \frac{9}{4}(x^2+y^4+z^6)^{\frac{4}{3}}.$$

Exam. 3. In like manner, the fluent of

$2x^2(\dot{x}y^2+xy\dot{y}+x^2\dot{x})\sqrt{(x^2+2y^2)}$, is $\frac{1}{3}(x^4+2x^2y^2)^{\frac{3}{2}}$.

21. The

21. The fluents of fluxions of the forms $\frac{x^n\dot{x}}{x \pm a}$, $\frac{x^n\dot{x}}{x^2 \pm a^2}$, &c, or $\frac{x^{cn-1}\dot{x}}{x^c \pm a}$, &c, where c and n are whole numbers, will be found in finite terms, by dividing the numerator by the denominator, using the variable letter x as the first term in the divisor, continuing the division till the powers of x are exhausted; after which, the last remainder will be the fluxion of a logarithm, or of a circular arc, &c.

Example 1. To find the fluent of $\frac{x\dot{x}}{a+x}$ or $\frac{x\dot{x}}{x+a}$.

By division, $\frac{x\dot{x}}{x+a} = \dot{x} - \frac{a\dot{x}}{x+a}$, where the remainder $\frac{a\dot{x}}{x+a}$ is evidently $= a \times$ the fluxion of the hyperbolic logarithm of $a + x$: therefore the whole fluent of the proposed fluxion is $x - a \times$ hyp. log. of $(a + x)$. In like manner it will be found that,

Ex. 2. The fluent of $\frac{x\dot{x}}{x-a}$, is $x + a \times$ hyp. log. of $(x - a)$.

Ex. 3. The fluent of $\frac{x\dot{x}}{a-x}$, is $- x - a \times$ hyp. log. of $(a - x)$.

Ex. 4. The fluent of $\frac{x^2\dot{x}}{a+x}$, is $\frac{1}{2}x^2 - ax + a^2 \times$ hyp. log. of $(a + x)$.

Ex. 5. The fluent of $\frac{x^2\dot{x}}{a-x}$, is $- \frac{1}{2}x^2 - ax - a^2 \times$ hyp. log. of $(a - x)$.

Ex. 6. The fluent of $\frac{x^2\dot{x}}{x-a}$, is $\frac{1}{2}x^2 + ax + a^2 \times$ hyp. log. of $(x - a)$.

Ex. 7. The fluent of $\frac{x^3\dot{x}}{x+a}$, is $\frac{1}{3}x^3 - \frac{1}{2}ax^2 + a^2x - a^3 \times$ hyp. log. of $(x + a)$.

Ex. 8. The fluent of $\frac{x^3\dot{x}}{x-a}$, is $\frac{1}{3}x^3 + \frac{1}{2}ax^2 + a^2x + a^3 \times$ hyp. log. of $(x - a)$.

Ex. 9. The fluent of $\frac{x^3\dot{x}}{a-x}$, is $- \frac{1}{3}x^3 - \frac{1}{2}ax^2 - a^2x + a^3 \times$ hyp. log. of $(a - x)$.

Ex. 10. The fluent of $\frac{x^4\dot{x}}{a+x}$, is $\frac{1}{4}x^4 - \frac{1}{3}ax^3 + \frac{1}{2}a^2x^2 - a^3x + a^4 \times$ hyp. log. $(a + x)$.

Ex. 11. The fluent of $\frac{x^n\dot{x}}{a+x}$, is $\frac{x^n}{n} - \frac{ax^{n-1}}{n-1} + \frac{a^2x^{n-2}}{n-2} - \frac{a^3x^{n-3}}{n-3}$ + &c $\pm a^n \times$ h. l. $(a + x)$.

Ex.

Ex. 12. The fluent of $\frac{x^n\dot{x}}{a-x}$, is $-\frac{x^n}{n}-\frac{ax^{n-1}}{n-1}-\frac{a^2x^{n-2}}{n-2}-\frac{a^3x^{n-3}}{n-3}$ &c $-\,a^n \times$ h. l. $(a-x)$.

Ex. 13. The fluent of $\frac{x^n\dot{x}}{x-a}$, is $\frac{x^n}{n}+\frac{ax^{n-1}}{n-1}+\frac{a^2x^{n-2}}{n-2}+\frac{a^3x^{n-3}}{n-3}$ &c $+\,a^n \times$ h. l. $(x-a)$.

Ex. 14. The fluent of $\frac{x^2\dot{x}}{x^2+a^2}=$ (by division) $\dot{x}-\frac{a^2\dot{x}}{a^2+x^2}$, is, (by form 11 vol. 2) x − cir. arc of radius a and tang. x, or $x-\frac{1}{2}a\times$ cir. arc of rad. 1 and cosine $\frac{a^2-x^2}{a^2+x^2}$. In like manner,

Ex. 15. The fluent of $\frac{x^2\dot{x}}{a^2-x^2}$, or of $-\dot{x}+\frac{a^2\dot{x}}{a^2-x^2}$, is $-x+\frac{1}{2}a\times$ h. l. $\frac{a+x}{a-x}$, by form 10. And

Ex. 16. The fluent of $\frac{x^2\dot{x}}{x^2-a^2}=x+\frac{a^2\dot{x}}{x^2-a^2}$, is $x+\frac{1}{2}a\times$ hyp. log. $\frac{x-a}{x+a}$, by the same form.

22. In like manner for the fluents of $\frac{x^4\dot{x}}{a^2\pm x^2}$. Thus,

Ex. 17. The fluent of $\frac{x^4\dot{x}}{a^2+x^2}=x^2\dot{x}-a^2\dot{x}+\frac{a^4\dot{x}}{a^2+x^2}$, is (by form 11), $\frac{1}{3}x^3-a^2x+a^2\times$ cir. arc to rad. a and tang. x, or $\frac{1}{3}x^3-a^2x+\frac{1}{2}a^3\times$ cir. arc to rad. 1 and cosine $\frac{a^2-x^2}{a^2+x^2}$. And

Ex. 18. The fluent of $\frac{x^4\dot{x}}{a^2-x^2}=-x^2\dot{x}-a^2\dot{x}+\frac{a^4\dot{x}}{a^2-x^2}$, is $-\frac{1}{3}x^3-a^2x+\frac{1}{2}a^3\times$ hyp. log. $\frac{a+x}{a-x}$, by form 10. Also

Ex. 19. The fluent of $\frac{x^4\dot{x}}{x^2-a^2}=x^2\dot{x}+a^2\dot{x}+\frac{a^4\dot{x}}{x^2-a^2}$, is $\frac{1}{3}x^3+a^2x+\frac{1}{2}a^3\times$ hyp. log. $\frac{x-a}{x+a}$, by form 10.

23. And in general for the fluent of $\frac{x^n\dot{x}}{x^2\pm a^2}$, where n is any even positive number, by dividing till the powers of x in the numerator are exhausted, the fluents will be found as before. And first for the denominator x^2+a^2, as in

Ex. 20. For the fluent of $\frac{x^n\dot{x}}{x^2+a^2}=$ (by actual division) $x^{n-2}\dot{x}-a^2x^{n-4}\dot{x}+a^4x^{n-6}-$ &c $\pm\,a^{n-2}\dot{x}\mp\frac{a^n\dot{x}}{x^2+a^2}$; the number of terms in the quotient being $\frac{1}{2}n$, and the remainder $\mp\frac{a^n\dot{x}}{x^2+a^2}$, viz, − or + according as that number of terms is odd or even. Hence, as before, the fluent

is $\frac{x^{n-1}}{n-1} - \frac{a^2x^{n-3}}{n-3} + \&c \ldots \pm a^{n-2}x \mp a^{n-2} \times$ arc to rad. a and tan x, or $\frac{x^{n-1}}{n-1} - \frac{a^2x^{n-3}}{n-3} + \&c \ldots \pm a^{n-2}x \mp \frac{1}{2}a^{n-1} \times$ arc to rad. 1 and cos. $\frac{a^2-x^2}{a^2+x^2}$.

Ex. 21. In like manner, the fluent of $\frac{x^n\dot{x}}{a^2-x^2}$, is $-\frac{x^{n-1}}{n-1} - \frac{a^2x^{n-3}}{n-3} - \frac{a^4x^{n-5}}{n-5} - \&c + \frac{1}{2}a^{n-1} \times$ hyp. log. $\frac{a+x}{a-x}$.

Ex. 22. And of $\frac{x^n\dot{x}}{x^2-a^2}$, is $\frac{x^{n-1}}{n-1} + \frac{a^2x^{n-3}}{n-3} + \&c + \frac{1}{2}a^{n-1} \times$ hyp. log. $\frac{x-a}{x+a}$.

24. In a similar manner we are to proceed for the fluents of $\frac{x^n\dot{x}}{a^2 \pm x^2}$, when n is any odd number, by dividing by the denominator inverted, till the first power of x only be found in the remainder, and when of course there will be one term less in the quotient than in the foregoing case, when n was an even number; but in the present case the log. fluent of the remainder will be found by the 8th form in the table of fluents in the 2d volume.

Ex. 22. Thus, for the fluent of $\frac{x^n\dot{x}}{x^2+a^2}$, where n is an odd number, the quotient by division as before, is $x^{n-2}\dot{x} - a^2x^{n-4}\dot{x} + a^4x^{n-6}\dot{x} - \&c \pm a^{n-3}x\dot{x}$, the number of terms being $\frac{n-1}{2}$, and the remainder $\mp \frac{a^{n-1}x\dot{x}}{x^2+a^2}$. Therefore the fluent is $\frac{x^{n-1}}{n-1} - \frac{a^2x^{n-3}}{n-3} + \&c \ldots \pm \frac{a^{n-3}x^2}{2} \mp \frac{1}{2}a^{n-1} \times$ h. l. $x^2 + a^2$.

Ex. 23. The fluent of $\frac{x^n\dot{x}}{x^2-a^2}$ is obtained in the same manner, and has the same terms, but the signs are all positive, and the remainder is $+ \frac{1}{2}a^{n-1} \times$ hyp. log. $x^2 - a^2$.

Ex. 24. Also the fluent of $\frac{x^n\dot{x}}{a^2-x^2}$ is still the same, but the signs are all negative, and the remainder is $- \frac{1}{2}a^{n-1} \times$ hyp. log. $a^2 - x^2$. Hence also,

Ex. 25. The fluent of $\frac{x^3\dot{x}}{x^2+a^2}$, is $\frac{1}{2}x^2 - \frac{1}{2}a^2 \times$ hyp. log. of $x^2 + a^2$.

Ex. 26. The fluent of $\frac{x^3\dot{x}}{x^2-a^2}$, is $\frac{1}{2}x^2 + \frac{1}{2}a^2 \times$ hyp. log. of $x^2 - a^2$.

Ex. 27. The fluent of $\frac{x^3\dot{x}}{a^2-x^2}$, is $-\frac{1}{2}x^2 - \frac{1}{2}a^2 \times$ hyp. log. of $a^2 - x^2$.

Ex.

Ex. 28. The fluent of $\frac{x^5\dot{x}}{x^2+a^2}$, is $\frac{1}{4}x^4 - \frac{1}{2}a^2x^2 + \frac{1}{2}a^4 \times$ hyp. log. $x^2 + a^2$.

Ex. 29. The fluent of $\frac{x^5\dot{x}}{x^2-a^2}$, is $\frac{1}{4}x^4 + \frac{1}{2}a^2x^2 + \frac{1}{2}a^4 \times$ hyp. log. $x^2 - a^2$.

Ex. 30. The fluent of $\frac{x^5\dot{x}}{a^2-x^2}$, is $-\frac{1}{4}x^4 - \frac{1}{2}a^2x^2 - \frac{1}{2}a^4 \times$ hyp. log. $a^2 - x^2$.

25. *Ex.* 31. In a similar manner may be found the fluents of $\frac{x^{cn-1}\dot{x}}{x^n \pm a^n}$, where c is any whole positive number, by dividing till the remainder be $\frac{a^{(c-1)n}x^{n-1}\dot{x}}{x^n \pm a^n}$, which can always be done, and the fluent of that remainder will be had by the 8th form in vol. 2. Thus, by dividing first by $x^n + a^n$, the terms are, $x^{cn-n-1}\dot{x} - a^nx^{cn-2n-1}\dot{x} + a^{2n}x^{cn-3n-1}\dot{x} - +$ &c till the last term be $a^{(d-1)n}x^{(c-d)n-1}$, and the remainder $\frac{a^{dn}x^{(c-d)n-1}\dot{x}}{x^n+a^n} = \frac{a^{(c-1)n}x^{n-1}\dot{x}}{x^n+a^n}$ when d is $= c-1$, or 1 less than c, which is also the number of the terms in the quotient; and therefore the fluent is

$$\frac{x^{cn-n}}{cn-n} - \frac{a^nx^{cn-2n}}{cn-2n} + \frac{a^{2n}x^{cn-3n}}{cn-3n} \ldots\ldots \pm \frac{a^{(c-2)n}x^n}{n} \mp \frac{1}{n}a^{(c-1)n} \times$$

hyp. log. of $x^n + a^n$. In like manner,

Ex. 32. The fluent of $\frac{x^{cn-1}\dot{x}}{x^n - a^n}$ has all the same terms as the former, but their signs all + or positive, and the remainder $\frac{1}{n}a^{(c-1)n} \times$ hyp. log. of $x^n - a^n$. Also in like manner

Ex. 33. The fluent of $\frac{x^{cn-1}\dot{x}}{a^n - x^n}$ has all the very same terms, but all negative, and the remainder $-\frac{1}{n}a^{(c-1)n} \times$ hyp. log. of $a^n - x^n$.

Ex. 34. The fluent of $\frac{x^{cn-1}\dot{x}}{b \pm cx^n} = \frac{1}{c} \times \frac{x^{cn-1}\dot{x}}{\frac{b}{c} \pm x^n}$ is also the same with the preceding, by substitut. $\frac{b}{c}$ for a^n, and multiplying the whole series by the fraction $\frac{1}{c}$.

26. When the numerator is compound, as well as the denominator, the expression may, in a similar manner by division, be reduced to like terms admitting of finite fluents. Thus, for

Ex.

Ex. 35. To find the fluent of $\frac{a-bx^2}{c+dx^2} \times x\dot{x} = \frac{ax\dot{x}-bx^3\dot{x}}{c+dx^2}$.

By division this becomes $-\frac{b}{d}x\dot{x} + \frac{ad+bc}{dd} \times \frac{x\dot{x}}{\frac{c}{d}+x^2}$; and its fluent $-\frac{b}{2d}x^2 + \frac{ad+bc}{2d^2} \times$ hyp. log. of $\frac{c}{d}+x^2$.

27. There are certain methods of finding fluents one from another, or of deducing the fluent of a proposed fluxion from another fluent previously known or found. There are hardly any general rules however that will suit all cases; but they mostly consist in assuming some quantity y in the form of a rectangle or product of two factors, which are such, that the one of them drawn into the fluxion of the other may be of the form of the proposed fluxion; then taking the fluxion of the assumed rectangle, there will thence be deduced a value of the proposed fluxion in terms that will often admit of finite fluents. The manner in such cases will better appear from the following examples.

Ex. 1. To find the fluent of $\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$.

Here it is obvious that if y be assumed $= x\sqrt{(x^2+a^2)}$, then one part of the fluxion of this product, viz, $x \times$ flux. of $\sqrt{(x^2+a^2)}$, will be of the same form as the fluxion proposed. Putting therefore the assumed rectangle $y = x\sqrt{(x^2+a^2)}$ into fluxions, it is $\dot{y} = \dot{x}\sqrt{(x^2+a^2)} + \frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$. But as the former part, viz, $\dot{x}\sqrt{(x^2+a^2)}$, does not agree with any of our preceding forms, which have been integrated, multiply it by $\sqrt{(x^2+a^2)}$, and subscribe the same as a denominator to the product, by which that part becomes

$\frac{x^2+a^2}{\sqrt{(x^2+a^2)}}\dot{x} = \frac{x^2\dot{x}+a^2\dot{x}}{\sqrt{(x^2+a^2)}}$; this united with the former part, makes the whole $\dot{y} = \frac{2x^2\dot{x}}{\sqrt{(x^2+a^2)}} + \frac{a^2\dot{x}}{\sqrt{(x^2+a^2)}}$; hence the given fluxion $\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{2}\dot{y} - \frac{1}{2}a^2 \times \frac{\dot{x}}{\sqrt{(x^2+a^2)}}$, and its fluent is therefore $\frac{1}{2}y - \frac{1}{2}a^2 \times \int\frac{\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{2}x\sqrt{(x^2+a^2)} - \frac{1}{2}a^2 \times$ hyp. log. of $x + \sqrt{(x^2+a^2)}$, by the 12th form of fluents.

Ex. 2. In like manner the fluent of $\frac{x^2\dot{x}}{\sqrt{(x^2-a^2)}}$ will be found from that of $\frac{\dot{x}}{\sqrt{(x^2-a^2)}}$ by the same 12th form, and is $= \frac{1}{2}x\sqrt{(x^2-a^2)} + \frac{1}{2}a^2 \times$ hyp. log. $x + \sqrt{(x^2-a^2)}$.

Ex. 3. Also in a similar manner, by the 13th form, the fluent

fluent of $\frac{x^2\dot{x}}{\sqrt{(a^2-x^2)}}$ will be found from that of $\frac{\dot{x}}{\sqrt{(a^2-x^2)}}$, and comes out $-\frac{1}{2}x\sqrt{(a^2-x^2)}+\frac{1}{2}a\times$ cir. arc to radius a and sine x.

Ex. 4. In like manner, the fluent of $\frac{x^4\dot{x}}{\sqrt{(x^2+a^2)}}$ will be found from that of $\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$. Here it is manifest that y must be assumed $= x^3\sqrt{(x^2+a^2)}$, in order that one part of its fluxion, viz, $\dot{x}\times$ flux. of $\sqrt{(x^2+a^2)}$ may agree with the proposed fluxion. Thus, by taking the fluxion, and reducing as before, the fluent of $\frac{x^4\dot{x}}{\sqrt{(x^2+a^2)}}$ will be found $=$ $\frac{1}{4}x^3\sqrt{(x^2+a^2)}-\frac{3}{4}a^2\times f\frac{x^2\dot{x}}{\sqrt{(x^2+a^2)}}$.

Ex. 5. Thus also the fluent of $\frac{x^4\dot{x}}{\sqrt{(x^2-a^2)}}$ is $\frac{1}{4}x^3\sqrt{(x^2-a^2)}$ $+\frac{3}{4}a^2\times f\frac{x^2\dot{x}}{\sqrt{(x^2-a^2)}}$.

Ex. 6. And the $f\frac{x^4\dot{x}}{\sqrt{(a^2-x^2)}}$, is $-\frac{1}{4}x^3\sqrt{(a^2-x^2)}+$ $\frac{3}{4}a^2\times f\frac{x^2\dot{x}}{\sqrt{(a^2-x^2)}}$.

In like manner the student may find the fluents of $\frac{x^6\dot{x}}{\sqrt{(x^2\pm a^2)}}$, $\frac{x^8\dot{x}}{\sqrt{(x^2\pm a^2)}}$, &c, to $\frac{x^n\dot{x}}{\sqrt{(x^2\pm a^2)}}$, where n is any even number, each from the fluent of that which immediately precedes it in the series, by substituting for y as before. Thus the fluent of $\frac{x^n\dot{x}}{\sqrt{(x^2+a^2)}}=\frac{1}{n}x^{n-1}\sqrt{(x^2+a^2)}-\frac{n-1}{n}$ $a^2\times f\frac{x^{n-2}\dot{x}}{\sqrt{(x^2+a^2)}}$.

28. In like manner we may proceed for the series of similar expressions where the index of the power of x in the numerator is some odd number.

Ex. 1. To find the fluent of $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$. Here assuming $y=x^2\sqrt{(x^2+a^2)}$, and taking the fluxion, one part of it will be similar to the fluxion proposed. Thus, $\dot{y}=2x\dot{x}$ $\sqrt{(x^2+a^2)}+\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$; hence at once the given fluxion $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}=\dot{y}-2x\dot{x}\sqrt{(x^2+a^2)}$; theref. the required fluent is $y-f\,.\,2x\dot{x}\sqrt{(x^2+a^2)}=x^2\sqrt{(x^2+a^2)}-\frac{2}{3}(x^2+a^2)^{\frac{3}{2}}$, by the 2d form of fluents.

Ex. 2. In like manner the fluent of $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$, is $x^2\sqrt{(x^2-a^2)}-\frac{2}{3}(x^2-a^2)^{\frac{3}{2}}$.

Ex.

Ex. 3. And the fluent of $\frac{x^3\dot{x}}{\sqrt{(a^2-x^2)}} = -x^2\sqrt{(a^2-x^2)} - \frac{2}{3}(a^2-x^2)^{\frac{3}{2}}$.

Ex. 4. To find the flu. of $\frac{x^5\dot{x}}{\sqrt{(x^2+a^2)}}$, from that of $\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$.

Here it is manifest we must assume $y = x^4\sqrt{(x^2+a^2)}$. This in fluxions and reduced gives $\dot{y} = \frac{5x^5\dot{x}}{\sqrt{(x^2+a^2)}} + \frac{4a^2x^3\dot{x}}{\sqrt{(x^2+a^2)}}$, and hence $\frac{x^5\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{5}\dot{y} - \frac{4a^2}{5} - \frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$; and the flu. is $\frac{1}{5}y - \frac{4}{5}a^2 \times f\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}} = \frac{1}{5}x^4\sqrt{(x^2+a^2)} - \frac{4}{5}a^2 \times f\frac{x^3\dot{x}}{\sqrt{(x^2+a^2)}}$, the fluent of the latter part being as in ex. 1, above.

In like manner the student may find the fluents of $\frac{x^5\dot{x}}{\sqrt{(x^2-a^2)}}$ and $\frac{x^5\dot{x}}{\sqrt{(a^2-x^2)}}$. He may then proceed in a similar way for the fluents of $\frac{x^7\dot{x}}{\sqrt{(x^2 \pm a^2)}}$, $\frac{x^9\dot{x}}{\sqrt{(x^2 \pm a^2)}}$, &c, $\frac{x^n\dot{x}}{\sqrt{(x^2 \pm a^2)}}$, where n is any odd number, viz, always by means of the fluent of each preceding term in the series.

29. In a similar manner may the process be for the fluents of the series of fluxions,

$$\frac{\dot{x}}{\sqrt{(a \pm x)}},\ \frac{x\dot{x}}{\sqrt{(a \pm x)}},\ \frac{x^2\dot{x}}{\sqrt{(a \pm x)}},\ \&c, \ldots \frac{x^n\dot{x}}{\sqrt{(a \pm x)}},$$

using the fluent of each preceding term in the series, as a part of the next term, and knowing that the fluent of the first term $\frac{\dot{x}}{\sqrt{a \pm x}}$ is given, by the 2d form of fluents, $= 2\sqrt{(a \pm x)}$, of the same sign as x.

Ex. 1. To find the fluent of $\frac{x\dot{x}}{\sqrt{(x+a)}}$, having given that of $\frac{\dot{x}}{\sqrt{(x+a)}} = 2\sqrt{(x+a)} = \text{A}$ suppose. Here it is evident we must assume $y = x\sqrt{(x+a)}$, for then its flux. $\dot{y} = \frac{\frac{1}{2}x\dot{x}}{\sqrt{(x+a)}} + \dot{x}\sqrt{(x+a)} = \frac{\frac{1}{2}x\dot{x}}{\sqrt{(x+a)}} + \frac{x\dot{x}}{\sqrt{(x+a)}} + \frac{a\dot{x}}{\sqrt{(x+a)}} = \frac{\frac{3}{2}x\dot{x}}{\sqrt{(x+a)}} + a\dot{\text{A}}$; hence $\frac{x\dot{x}}{\sqrt{(x+a)}} = \frac{2}{3}\dot{y} - \frac{2}{3}a\dot{\text{A}}$; and the required fluent is $\frac{2}{3}y - \frac{2}{3}a\text{A} = \frac{2}{3}x\sqrt{(x+a)} - \frac{4}{3}a\sqrt{(x+a)} = (x-2a) \times \frac{2}{3}\sqrt{(x+a)}$.

In like manner the student will find the fluents of $\frac{x\dot{x}}{\sqrt{(x-a)}}$ and $\frac{x\dot{x}}{\sqrt{(a-x)}}$.

Ex. 2. To find the fluent of $\frac{x^2\dot{x}}{\sqrt{(x+a)}}$, having given that of $\frac{x\dot{x}}{\sqrt{(x+a)}} = \dot{\text{B}}$. Here y must be assumed $= x^2\sqrt{(x+a)}$; for then taking the flu. and reducing, there is found $\frac{x^2\dot{x}}{\sqrt{(x+a)}} = \frac{2}{5}\dot{y} -$

$\frac{2}{5}\dot{y} - \frac{4}{5}a\text{B}$; theref. $f\frac{x^2\dot{x}}{\sqrt{(x+a)}} = \frac{2}{5}y - \frac{4}{5}a\text{B} = \frac{2}{5}x^2\sqrt{(x+a)} - \frac{4}{5}a\text{B} = \frac{2}{5}x^2\sqrt{(x+a)} - \frac{4}{5}a(x-2a) \times \frac{2}{3}\sqrt{(x+a)} = (9x^2 - 4ax + 8a^2) \times \frac{2}{15}\sqrt{(x+a)}$.

In the same manner the student will find the fluents of $\frac{x^2\dot{x}}{\sqrt{(x-a)}}$ and of $\frac{x^2\dot{x}}{\sqrt{(a-x)}}$. And in general, the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{(x+a)}}$ being given $=$ C, he will find the fluent of $\frac{x^n\dot{x}}{\sqrt{(x+a)}} = \frac{2}{2n+1}x^n\sqrt{(x+a)} - \frac{2n}{2n+1}a\text{C}$.

30. In a similar way we might proceed to find the fluents of other classes of fluxions by means of other fluents in the table of forms in volume 2; as, for instance, such as $x\dot{x}\sqrt{(dx-x^2)}$, $x^2\dot{x}\sqrt{(dx-x^2)}$, $x^3\dot{x}\sqrt{(dx-x^2)}$, &c, depending on the fluent of $\dot{x}\sqrt{(dx-x^2)}$, the fluent of which, by the 16th tabular form, is the circular semisegment to diameter d and versed sine x, or the half or trilineal segment contained by an arc with its right sine and versed sine, the diameter being d.

Ex. 1. Putting then said semiseg. or fluent of $\dot{x}\sqrt{(dx-x^2)} = \text{A}$, to find the fluent of $x\dot{x}\sqrt{(dx-x^2)}$. Here assuming $y = (dx-x^2)^{\frac{3}{2}}$, and taking the fluxions, they are, $\dot{y} = \frac{3}{2}(d\dot{x} - 2x\dot{x})\sqrt{(dx-x^2)}$, hence $x\dot{x}\sqrt{(dx-x^2)} = \frac{1}{2}d\dot{x}\sqrt{(dx-x^2)} - \frac{1}{3}\dot{y} = \frac{1}{2}d\dot{\text{A}} - \frac{1}{3}\dot{y}$; therefore the required fluent, $fx\dot{x}\sqrt{(dx-x^2)}$, is $\frac{1}{2}d\text{A} - \frac{1}{3}y = \frac{1}{2}d\text{A} - \frac{1}{3}(dx-x^2)^{\frac{3}{2}} = \text{B}$ suppose.

Ex. 2. To find the fluent of $x^2\dot{x}\sqrt{(dx-x^2)}$, having that of $x\dot{x}\sqrt{(dx-x^2)}$ given $=$ B. Here assuming $y = x(dx-x^2)$, then taking the fluxions, and reducing, there results $\dot{y} = (\frac{5}{2}dx\dot{x} - 4x^2\dot{x})\sqrt{(dx-x^2)}$; hence $x^2\dot{x}\sqrt{(dx-x^2)} = \frac{5}{8}dx\dot{x}\sqrt{(dx-x^2)} - \frac{1}{4}\dot{y} = \frac{5}{8}d\dot{\text{B}} - \frac{1}{4}\dot{y}$, the flu. theref. of $x^2\dot{x}\sqrt{(dx-x^2)}$ is $\frac{5}{8}d\text{B} - \frac{1}{4}y = \frac{5}{8}d\text{B} - \frac{1}{4}x(dx-x^2)^{\frac{3}{2}}$.

Ex. 3. In the same manner the series may be continued to any extent; so that in general, the flu. of $x^{n-1}\sqrt{(dx-x^2)}$ being given $=$ C, then the next, or the flu. of $x^n\dot{x}\sqrt{(dx-x^2)}$ will be $\frac{2n+1}{n+2} \cdot \frac{1}{2}d\text{C} - \frac{1}{n+2}x^{n-1}(dx-x^2)^{\frac{3}{2}}$.

31. To find the fluent of such expressions as $\frac{\dot{x}}{\sqrt{(x^2 \pm 2ax)}}$, a case not included in the table of forms in vol. 2.

Put the proposed radical $\sqrt{(x^2 \pm 2ax)} = z$, or $x^2 \pm 2ax = z^2$; then, completing the square, $x^2 \pm 2ax + a^2 = z^2 + a^2$, and the root is $x \pm a = \sqrt{(z^2 + a^2)}$. The fluxion of this is $\dot{x} = \frac{z\dot{z}}{\sqrt{(z^2+a^2)}}$; theref. $\frac{\dot{x}}{\sqrt{(x^2 \pm 2ax)}} = \frac{\dot{z}}{\sqrt{(z^2+a^2)}}$; the fluent of

of which, by the 12th form, is the hyp. log. of $z + \sqrt{(z^2 + a^2)}$ = hyp. log. of $x \pm a + \sqrt{(x^2 \pm 2ax)}$, the fluent required.

Ex. 2. To find now the fluent of $\frac{x\dot{x}}{\sqrt{(x^2 + 2ax)}}$, having given, by the above example, the fluent of $\frac{\dot{x}}{\sqrt{(x^2 + 2ax)}} = \text{A}$ suppose. Assume $\sqrt{(x^2 + 2ax)} = y$; then its fluxion is $\frac{x\dot{x} + a\dot{x}}{\sqrt{(x^2 + 2ax)}} = \dot{y}$; theref. $\frac{x\dot{x}}{\sqrt{(x^2 + 2ax)}} = \dot{y} - \frac{a\dot{x}}{\sqrt{(x^2 + 2ax)}} = \dot{y} - a\dot{\text{A}}$; the fluent of which is $y - a\text{A} = \sqrt{(x^2 + 2ax)} - a\text{A}$, the fluent sought.

Ex. 3. Thus also, this fluent of $\frac{x\dot{x}}{\sqrt{(x^2+2ax)}}$ being given, the flu. of the next in the series, or $\frac{x^2\dot{x}}{\sqrt{(x^2 + 2ax)}}$ will be found, by assuming $x\sqrt{(x^2 + 2ax)} = y$; and so on for any other of the same form. As, if the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{(x^2 + 2ax)}}$ be given $= \text{C}$; then, by assuming $x^{n-1}\sqrt{(x^2 + 2ax)} = y$, the fluent of $\frac{x^n\dot{x}}{\sqrt{(x^2 + 2ax)}} = \frac{1}{n} x^{n-1}\sqrt{(x^2 + 2ax)} - \frac{2n-1}{n}a\text{C}$.

Ex. 4. In like manner, the fluent of $\frac{\dot{x}}{\sqrt{(x^2 - 2ax)}}$ being given, as in the first example, that of $\frac{x\dot{x}}{\sqrt{(x^2 - 2ax)}}$ may be found; and thus the series may be continued exactly as in the 3d ex. only taking $-2ax$ for $+2ax$.

32. Again, having given the fluent of $\frac{\dot{x}}{\sqrt{(2ax - x^2)}}$, which, by pa. 321 vol. 2, is $\frac{1}{a} \times$ circular arc to radius a and versed sine x, the fluents of $\frac{x\dot{x}}{\sqrt{(2ax - x^2)}}$, $\frac{x^2\dot{x}}{\sqrt{(2ax - x^2)}}$, &c . . $\frac{x^n\dot{x}}{\sqrt{(2ax - x^2)}}$, may be assigned by the same method of continuation. Thus,

Ex. 1. For the fluent of $\frac{x\dot{x}}{\sqrt{(2ax - x^2)}}$, assume $\sqrt{(2ax - x^2)} = y$; the required fluent will be found $= -\sqrt{(2ax - x^2)} + \text{A}$ or arc to radius a and vers. x.

Ex. 2. In like manner the fluent of $\frac{x^2\dot{x}}{\sqrt{(2ax - x^2)}}$ is $f\frac{\frac{3}{2}ax\dot{x}}{\sqrt{(2ax - x^2)}} - \frac{1}{2}x\sqrt{(2ax - x^2)} = \frac{3}{2}a\text{A} - \frac{3a + x}{2}\sqrt{(2ax - x^2)}$, where A denotes the arc mentioned in the last example.

Ex. 3. And in general the fluent of $\frac{x^n\dot{x}}{\sqrt{(2ax - x^2)}}$ is $\frac{2n-1}{n}a\text{C} - \frac{1}{n}x^{n-1}\sqrt{(2ax - x^2)}$, where C is the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{(2ax - x^2)}}$, the next preceding term in the series.

33. Thus

33. Thus also, the fluent of $\dot{x}\sqrt{(x-a)}$ being given, $= \frac{2}{3}(x-a)^{\frac{3}{2}}$, by the 2d form, the fluents of $x\dot{x}\sqrt{(x-a)}$, $x^2\dot{x}\sqrt{(x-a)}$, &c . . . $x^n\dot{x}\sqrt{(x-a)}$, may be found. And in general, if the fluent of $x^{n-1}\dot{x}\sqrt{(x-a)} = c$ be given; then by assuming $x^n(x-a)^{\frac{3}{2}} = y$, the fluent of $x^n\dot{x}\sqrt{(x-a)}$ is found $= \frac{2}{2n+3}x^n(x-a)^{\frac{3}{2}} + \frac{2na}{2n+3}c$.

34. Also, given the fluent of $(x-a)^m\dot{x}$, which is $\frac{1}{m+1}(x-a)^{m+1}$ by the 2d form, the fluents of the series $(x-a)^m x\dot{x}$, $(x-a)^m x^2\dot{x}$, &c . . . $(x-a)^m x^n\dot{x}$ can be found. And in general, the fluent of $(x-a)^m x^{n-1}\dot{x}$ being given $= c$; then by assuming $(x-a)^{m+1}x^n = y$, the fluent of $(x-a)^m x^n\dot{x}$ is found $= \frac{x^n(x-a)^{m+1}+nac}{m+n+1}$.

Also, by the same way of continuation, the fluents of $x^n\dot{x}\sqrt{(a \pm x)}$ and of $x^n\dot{x}(a \pm x)^m$ may be found.

35. When the fluxional expression contains a trinomial quantity, as $\sqrt{(b+cx+x^2)}$, this may be reduced to a binomial, by substituting another letter for the unknown one x, connected with half the coefficient of the middle term with its sign. Thus, put $z = x + \frac{1}{2}c$: then $z^2 = x^2 + cx + \frac{1}{4}c^2$; theref. $z^2 - \frac{1}{4}c^2 = x^2 + cx$, and $z^2 + b - \frac{1}{4}c^2 = x^2 + cx + b$ the given trinomial; which is $= z^2 + a^2$, by putting $a^2 = b - \frac{1}{4}c^2$.

Ex. 1. To find the fluent of $\frac{3\dot{x}}{\sqrt{(5+4x+x^2)}}$.

Here $z = x + 2$; then $z^2 = x^2 + 4x + 4$, and $z^2 + 1 = 5 + 4x + x^2$, also $\dot{x} = \dot{z}$; theref. the proposed fluxion reduces to $\frac{3\dot{z}}{\sqrt{(1+z^2)}}$; the fluent of which, by the 12th form in the 2d vol. is 3 hyp. log. of $z + \sqrt{(1+z)} = 3$ hyp. log. $x + 2 + \sqrt{(5+4x+x^2)}$.

Ex. 2. To find the fluent of $\dot{x}\sqrt{(b+cx+dx^2)} = \dot{x}\sqrt{d} \times \sqrt{(\frac{b}{d} + \frac{c}{d}x + x^2)}$.

Here assuming $x + \frac{c}{2d} = z$; then $\dot{x} = \dot{z}$, and the proposed flux. reduces to $\dot{z}\sqrt{d} \times \sqrt{(z^2 + \frac{b}{d} - \frac{c^2}{4d^2})} = \dot{z}\sqrt{d} \times \sqrt{(z^2+a^2)}$, putting a^2 for $\frac{b}{d} - \frac{c^2}{4d^2}$; and the fluent will be found by a similar process to that employed in ex. 1 art. 27.

Ex. 3. In like manner, for the flu. of $x^{n-1}\dot{x}\sqrt{(b+cx^n+dx^{2n})}$, assuming $x^n + \frac{c}{2d} = z$, $nx^{n-1}\dot{x} = \dot{z}$, and $x^{n-1}\dot{x} = \frac{1}{n}\dot{z}$;

hence

hence $x^{2n} + \frac{c}{d}x^n + \frac{c^2}{4d^2} = z^2$, and $\sqrt{(dx^{2n} + cx^n + b)} = \sqrt{d} \times \sqrt{(x^{2n} + \frac{c}{d}x^n + \frac{b}{d})} = \sqrt{d} \times \sqrt{(z^2 + \frac{b}{d} - \frac{c^2}{4d^2})} = \sqrt{d} \times \sqrt{(z^2 \pm a^2)}$, putting $\pm a^2 = \frac{b}{d} - \frac{c^2}{4d^2}$; hence the given fluxion becomes $\frac{1}{n}\dot{z}\sqrt{d} \times \sqrt{(z^2 \pm a^2)}$, and its fluent as in the last example.

Ex. 4. Also, for the fluent of $\frac{x^{n-1}\dot{x}}{b + cx^n + dx^{2n}}$; assume $x^n + \frac{c}{2d} = z$, then the fluxion may be reduced to the form $\frac{1}{dn} \times \frac{\dot{z}}{z^2 \pm a^2}$, and the fluent found as before.

So far on this subject may suffice on the present occasion. But the student who may wish to see more on this branch, may profitably consult Mr. Dealtry's very methodical and ingenious treatise on Fluxions, lately published, from which several of the foregoing cases and examples have been taken or imitated.

CHAPTER XI.

ON THE MOTION OF MACHINES, AND THEIR MAXIMUM EFFECTS.

ART. 1. When forces acting in contrary directions, or in any such directions as produce contrary effects, are applied to machines, there is, with respect to every simple machine (and of consequence with respect to every combination of simple machines) a certain relation between the powers and the distances at which they act, which, if subsisting in any such machine when at rest, will always keep it in a state of rest, or of *statical* equilibrium; and for this reason, because the efforts of these powers, when thus related, with regard to magnitude and distance, being equal and opposite, annihilate each other, and have no tendency to change the state of the system to which they are applied. So also, if the same machine have been put into a state of *uniform* motion, whether rectilinear or rotatory, by the action of any power distinct from those we are now considering, and these two powers be made to act upon the machine in such motion in a similar manner to that in which they acted upon it when at rest, their simultaneous action will preserve it in that state of

of uniform motion, or of *dynamical* equilibrium; and this for the same reason as before, because their contrary effects destroy each other, and have therefore no tendency to change the *state* of the machine. But, if at the time a machine is in a state of balanced rest, any one of the opposite forces be increased while it continues to act at the same distance, this excess of force will disturb the statical equilibrium, and produce motion in the machine; and if the same excess of force continues to act in the same manner, it will, like every constant force, produce an accelerated motion; or, if it should undergo particular modifications when the machine is in different positions, it may occasion such variations in the motion as will render it alternately accelerated and retarded. Or the different species of resistance to which a moving machine is subjected, as the rigidity of ropes, friction, resistance of the air, &c, may so modify a motion, as to change a regular or irregular variable motion into one which is uniform.

2. Hence then the motion of machines may be considered as of *three* kinds. 1. That which is gradually accelerated, which obtains commonly in the first instants of the communication. 2. That which is entirely uniform. 3. That which is alternately accelerated and retarded. Pendulum clocks, and machines which are moved by a balance, are related to the third class. Most other machines, a short time after their motion is commenced, fall under the second. Now though the motion of a machine is alternately accelerated and retarded, it may, notwithstanding, be measured by an uniform motion, because of the periodical and regular repetition which may exist in the acceleration and retardation. Thus the motion of a second's pendulum, considered in respect to a single oscillation, is accelerated during the first half second, and retarded during the next: but the same motion taken for many oscillations may be considered as uniform. Suppose, for example, that the extent of each oscillation is 5 inches, and that the pendulum has made 10 oscillations: its total effect will be to have run over 50 inches in 10 seconds; and, as the space described in each second is the same, we may compare the effect to that produced by a moveable which moves for 10 seconds with a velocity of 5 inches per second. We see, therefore, that the theory of machines whose motions are uniform, conduces naturally to the estimation of the effects produced by machines whose motion is alternately accelerated and retarded: so that the problems comprised in this chapter will be directed to those machines whose motions fall under the first two heads; such problems being of far the greatest utility in practice.

Defs. 1. When in a machine there is a system of forces or of powers mutually in opposition, those which produce or tend to produce a certain effect are called *movers* or *moving powers*; and those which produce or tend to produce an effect which opposes those of the moving powers, are called *resistances*. If various movers act at the same time, their equivalent (found by means of prop. 7, Motion and Forces) is called individually *the moving force*; and, in like manner, the resultant of all the resistances reduced to some one point, *the resistance*. This reduction in all cases simplifies the investigation.

2. The *impelled point* of a machine is that to which the action of the moving power may be considered as immediately applied; and the *working point* is that where the resistance arising from the work to be performed immediately acts, or to which it ought all to be reduced. Thus, in the wheel and axle, (Mechan. prop. 32), where the moving power P is to overcome the weight or resistance W, by the application of the cords to the wheel and to the axle, B is the impelled point, and A the working point.

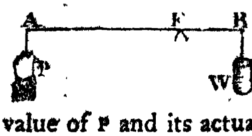
3. The *velocity of the moving power* is the same as the velocity of the impelled point; the *velocity of the resistance* the same as that of the working point.

4. The *performance* or *effect* of a machine, or the *work done*, is measured by the product of the resistance into the velocity of the working point; the *momentum of impulse* is measured by the product of the moving force into the velocity of the impelled point.

These definitions being established, we may now exhibit a few of the most useful problems, giving as much variety in their solutions as may render one or other of the methods of easy application to any other cases which may occur.

PROPOSITION I.

If R *and* r *be the distances of the power* P, *and the weight or resistance* W, *from the fulcrum* F *of a straight lever; then will the velocity of the power and of the weight at the end of any time* t *be* $\frac{R^2P - RrW}{R^2P + r^2W}gt$, *and* $\frac{RrP - r^2W}{R^2P + r^2W}gt$, *respectively, the weight and inertia of the lever itself not being considered.*



If the effort of the power balanced that of the resistance, P would be equal to $\frac{rW}{R}$. Consequently, the difference between this value of P and its actual value, or $P - \frac{r}{R}W$, will be the force which tends to move the

the lever. And because this power applied to the point A accelerates the masses P and W, the mass to be substituted for W, in the point A, must be $\frac{r^2}{R^2}$W, (Mechan. prop. 50) in order that this mass at the distance R may be equally accelerated with the mass W at the distance R. Hence the power $P - \frac{r}{R}W$ will accelerate the quantity of matter $P + \frac{r^2}{R^2}W$; and the accelerating force $F = (P - \frac{r}{R}W) \div (P + \frac{r^2}{R^2}W) = \frac{PR^2 - RrW}{PR^2 + r^2W}$. But (vol. ii. p. 335) $v \propto Ft$ or is $= gtF$ (g being $= 32\frac{1}{6}$ feet); which in this case $= \frac{P - RrW}{R^2P + r^2W} \cdot gt$, the velocity of P. And because veloc. of P : veloc. of W :: R : r, therefore veloc. of $W = \frac{r}{R}$ veloc. of $P = \frac{r}{R} \times \frac{R^2P - RrW}{R^2P + r^2W}gt = \frac{RrP - r^2W}{R^2P + r^2W} \cdot gt$.

Cor. 1. The space described by the power in the time t, will be $= \frac{R^2P - RrW}{R^2P + r^2W} \cdot \frac{1}{2}gt^2$; the space described by W in the same time will be $= \frac{RrP - r^2W}{R^2P + r^2W} \cdot \frac{1}{2}gt^2$.

Cor. 2. If R : r :: n : 1, then will the force which accelerates A be $= \frac{Pn^2 - Wn}{Pn^2 + W}$.

Cor. 3. If at the same time the inertia of the moving force P be = 0, as in muscular action, the force accelerating A will be $= \frac{Pn^2 - Wn}{W}$.

Cor. 4. If the mass moved have no weight, but possesses inertia only, as when a body is moved along a horizontal plane, the force which accelerates A will be $= \frac{Pn^2}{Pn^2 + W}$. And either of these values may be readily introduced into the investigation.

Cor. 5. The work done in the time t, if we retain the original notation, will be $= \frac{RrP - r^2W}{R^2P + r^2W}gt \times W = \frac{RrPW - r^2W^2}{R^2P + r^2W} \cdot gt$.

Cor. 6. When the work done is to be a maximum, and we wish to know the weight when P is given, we must make the fluxion of the last expression = 0. Then we shall have $rR^3P^2 - 2r^2R^2PW - r^4W^2 = 0$ and $W = P \times [\sqrt{(\frac{R^4}{r^4} + \frac{R^3}{r^3})} - \frac{R^2}{r^2}]$.

Cor. 7. If R : r :: n : 1, the preceding expression will become $W = P \times [\sqrt{(n^4 + n^3)} - n^2]$.

Cor. 8. When the arms of the lever are equal in length, that is, when $n = 1$, then is $W = P \times (\sqrt{2} - 1) = \cdot414214P$, or nearly $\frac{5}{12}$ of the moving force.

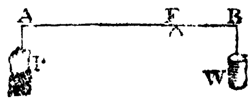
 Scholium.

Scholium.

If we in like manner investigate the formulæ relating to motion on the axis in peritrochio, it will be seen that the expressions correspond exactly. Hence it follows, that when it is required to proportion the power and weight so as to obtain a maximum effect on the wheel and axle, (the weight of the machinery not being considered), we may adopt the conclusions of cors. 6 and 7 of this prop. And in the extreme case where the wheel and axle becomes a pulley, the expression in cor. 8 may be adopted. The like conclusions may be applied to machines in general, if R and r represent the distances of the impelled and working points from the axis of motion; and if the various kinds of resistance arising from friction, stiffness of ropes, &c, be properly reduced to their equivalents at the working points, so as to be comprehended in the character W for resistance overcome.

PROPOSITION II.

Given R *and* r, *the arms of a straight lever,* M *and* m *their respective weights, and* P *the power acting at the extremity of the arm* R; *to find the weight raised at the extremity of the other arm when the effect is a maximum.*



In this case $\frac{1}{2}m$ is the weight of the shorter end reduced to B, and conseq. $\frac{mr}{2R}$ is the weight which, applied at A, would balance the shorter end: therefore $\frac{mr}{2R} + \frac{r}{R}W$, would sustain both the shorter end and the weight W in equilibrio. But $P + \frac{1}{2}M$ is the power really acting at the longer end of the lever; consequently $P + \frac{1}{2}M - (\frac{mr}{2R} + \frac{r}{R}W)$, is the absolute *moving* power. Now the distance of the centre of gyration of the beam from F*

---

* The distance of R, the centre of gyration, from C the centre or axis of motion, in some of the most useful cases, is as below:

In a circular wheel of uniform thickness . . . . . . $CR = \text{rad.} \sqrt{\frac{1}{2}}$.
In the periphery of a circle revolving about the diam. $CR = \text{rad} \sqrt{\frac{1}{2}}$.
In the plane of a circle . . . . ditto . . . . . $CR = \frac{1}{2}\text{rad.}$
In the surface of a sphere . . . ditto . . . . . $CR = \text{rad.} \sqrt{\frac{2}{3}}$.
In a solid sphere . . . . ditto . . . . . $CR = \text{rad.} \sqrt{\frac{2}{5}}$.
In a plane ring formed of circles whose radii are R, r, revolving about centre . . . . $CR = \sqrt{\frac{R^4}{2R^2 - 2r^2}}$
In a cone revolving about its vertex . . . . $CR = \frac{1}{2}\sqrt{\frac{12}{5}a^2 + \frac{3}{5}r^2}$
In a cone . . . . . . its axis . . . . $CR = r\sqrt{\frac{3}{10}}$
In a straight lever whose arms are R and r . . . . . $CR = \sqrt{\frac{R^3 + r^3}{3(R+r)}}$.

is

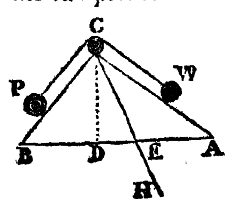
is $= \sqrt{\frac{R^3+r^3}{3(R+r)}}$, which let be denoted by ρ; then (Mechan. prop. 50) $\frac{\rho^2}{R^2} . (M+m)$ will represent the mass equivalent to the beam or lever when reduced to the point A; while the weight equivalent to W, when referred to that point, will be $\frac{r^2}{R^2}W$. Hence, proceeding as in the last prop. we shall have $\frac{\rho^2}{R^2} . (M+m) + P + \frac{r^2}{R^2}W$ for the inertia to be overcome; and $(P + \frac{1}{2}M - \frac{mr}{2R} - \frac{r}{R}W) \div \frac{\rho^2}{R^2}(M+m) + P + \frac{r^2}{R^2}W$ = the accelerating force of P, or of W reduced to A. Multiply this by W; and, for the sake of simplifying the process, put q for $P + \frac{1}{2}M - \frac{rm}{2R}$, and n for $P + \frac{\rho^2}{R^2}(M+m)$, then will $\frac{qW - \frac{rW^2}{R}}{n + \frac{r^2}{R^2}W}$ be a quantity which varies as the effect varies, and which, indeed, when multiplied by gt, denotes the effect itself. Putting the fluxion of this equal to nothing, and reducing, we at length find

$$W = \frac{R}{r}\sqrt{\left(\frac{nqR}{r} + \frac{n^2R^2}{r^2}\right)} - \frac{nR^2}{r^2}.$$

Cor. When $R = r$, and $M = m$, if we restore the values of n and q, the expression will become $W = \sqrt{(2P^2 + 2mP + \frac{4}{9}m^2)} - (P + \frac{2}{3}m)$.

PROPOSITION III.

Given the length l and angle e of elevation of an inclined plane BC; *to find the length* L *of another inclined plane* AC, *along which a given weight* W *shall be raised from the horizontal line* AB *to the point* C, *in the least time possible, by means of another given weight* P *descending along the given plane* CB *the two weights being connected by an inextensible thread* PCW *running always parallel to the two planes.*



Here we must, as a preliminary to the solution of this proposition, deduce expressions for the motion of bodies connected by a thread, and running upon double inclined planes. Let the angle of elevation CAD be E, while e is the elevation CBD. Then at the end of the time t, P will have a velocity v; and gravity would impress upon it, in the instant t following, a new velocity $= g \sin e . t$, provided

vided the weight P were then entirely free: but, by the disposition of the system, $\dot{v}$ will be the velocity which obtains in reality. Then, estimating the spaces in the direction CB, as the body W moves with an equal velocity but in a contrary sense, it is obvious that, by applying the 3d Law of Motion, the decomposition may be made as follows. At the end of the time $t + \dot{t}$ we have, for the velocity impressed on,

$$\text{P} \ldots v + g \sin e \,.\, \dot{t}, \text{ where } \begin{cases} v + \dot{v} \ldots\ldots & \text{effective veloc. from C towards B.} \\ g \sin e \,.\, \dot{t} - \dot{v} \ldots\ldots & \text{velocity destroyed.} \end{cases}$$

$$\text{W} \,.\, -v + g \sin \text{E} \,.\, \dot{t}, \text{ where } \begin{cases} -v - \dot{v} \ldots & \text{effective veloc. from C towards A.} \\ \dot{v} + g \sin \text{E} \,.\, \dot{t}, \ldots\ldots & \text{velocity destroyed.} \end{cases}$$

If, therefore, gravity impresses, during the time $\dot{t}$, upon the masses P, W, the respective velocities $g \sin e \,.\, \dot{t} - \dot{v}$, and $g \sin \text{E} \,.\, \dot{t} + \dot{v}$, the system will be in equilibrio. The quantities of motion being therefore equal, it will be

$$\text{P}g \sin e \,.\, \dot{t} - \text{P}\dot{v} = \text{W}g \sin \text{E} \,.\, \dot{t} + \text{W}\dot{v}.$$

Whence the effective accelerating force is found, i. e.

$$\varphi = \frac{\dot{v}}{\dot{t}} = \frac{\text{P} \sin e - \text{W} \sin \text{E}}{\text{P} + \text{W}} \times g.$$

Thus it appears that the motion is uniformly varied, and we readily find the equations for the velocity and space from which the conditions of the motion are determined: viz,

$$v = \frac{\text{P} \sin e - \text{W} \sin \text{E}}{\text{P} + \text{W}} \ldots s = \frac{\text{P} \sin e - \text{W} \sin \text{E}}{\text{P} + \text{W}} \,.\, \tfrac{1}{2} g t^2.$$

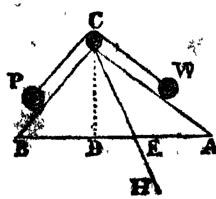
The latter of these two equations gives $t^2 = \dfrac{s(\text{P} + \text{W})}{\frac{1}{2}g(\text{P} \sin e - \text{W} \sin e)}$. But in the triangle ABC it is AC : BC : : sin B : sin A, that is, L : l : : sin e : sin E; hence $\frac{1}{m}\text{L} = \sin e$, and $\frac{1}{m} l = \sin \text{E}$; m being a constant quantity always determinable from the data given. And t^2 becomes $\dfrac{s(\text{P} + \text{W})}{\frac{1}{2}g \frac{1}{m}(\text{PL} - \text{W}l)}$. Now when any quantity, as t, is a minimum, its square is manifestly a minimum: so that substituting for s its equal L, and striking out the constant factors, we have $\dfrac{\text{L}^2}{\text{PL} - \text{W}l}$ = a min. or its fluxion $\dfrac{2\text{L}\dot{\text{L}}(\text{PL} - \text{W}l) - \text{PL}^2\dot{\text{L}}}{(\text{PL} - \text{W}l)^2} = 0$. Here, as in all similar cases, since the fraction vanishes, its numerator must be equal to 0; consequently $2\text{PL}^2 - 2\text{W}l\text{L} - \text{PL}^2 = 0$, $\text{PL} = 2\text{W}l$, or L : l : : 2W : P.

Cor. 1. Since neither sin e nor sin E enters the final equation, it follows, that if the elevation of the plane BC is not given, the problem is unlimited.

Cor.

Cor. 2. When $\sin e = 1$, BC coincides with the perpendicular CD, and the power P acts with all its intensity upon the weight W.* This is the case of the present problem which has commonly been considered.

Scholium.



This proposition admits of a neat *geometrical* demonstration. Thus, let CE be the plane upon which, if W were placed, it would be sustained in equilibrio by the power P on the plane CB, or the power P′ hanging freely in the vertical CD; then (Mechan. prop. 23) BC : CD : CE :: P : P′ : W. But W is to the force with which it tends to descend along the plane CA, as CA to CD; consequently, the weight P′ is to that force, as CA : CE; or the weight P on the plane BC, is to the same force in the same ratio; because either of these weights in their respective positions would sustain W on CE. Therefore the excess of P above that force (which excess is the power accelerating the motions of P and W) is to P, as CA + CE to CA; or, taking CH = CA, as EH to CA. Now, the motion being uniformly accelerated, we have $S \propto FT^2$, or $T^2 \propto \frac{S}{F}$: consequently, the square of the time in which AC is described by W, will be as AC directly, and as $\frac{EH}{AC}$ inversely; and will be least when $\frac{CA^2}{EH}$ is a *minimum;* that is when $\frac{CE^2}{EH} + EH + 2CE$, or (because 2CE is invariable) when $\frac{CE^2}{EH} + EH$ is a minimum. Now, as, when the sum of two quantities is given, their product is a *maximum* when they are equal to each other; so it is manifest that when their product is given, their sum must be a *minimum* when they are equal. But the product of $\frac{CE^2}{EH}$ and EH is CE^2, and consequently given; therefore the sum of $\frac{EC^2}{EH}$ and EH is least, when those parts are equal; that is, when EH = CE, or CA = 2CE. So that the length of the plane CA is double the length of that on which the weight W would be kept in equilibrio by P acting along CB.

When CD and CB coincide, the case becomes the same as that considered by Maclaurin, in his *View of Newton's Philosophical Discoveries*, pa. 183, 8vo. edit.

PROPOSITION IV.

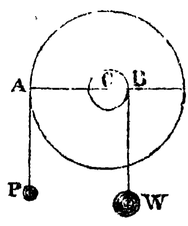
Let the given weight P *descend along* CB, *and by means of the thread* PCW *(running parallel to the planes) draw a weight* W *up the plane* AC: *it is required to find the value of* W, *when its momentum is a maximum, the lengths and positions of the planes being given. (See the preceding fig.).*

The general expression for the vel. is $v = \frac{\text{P} . \sin e - \text{W} \sin \text{E}}{\text{P} + \text{W}} gt$, which, by substitut. $\frac{1}{m}\text{L}$ for $\sin e$, and $\frac{1}{m} l$ for $\sin \text{E}$, becomes $v = \frac{\frac{1}{m}(\text{PL} - \text{W}l)}{\text{P} + \text{W}} gt$. This mul. into W, gives $\frac{\frac{1}{m}(\text{PWL} - \text{W}^2 l)}{\text{P} + \text{W}} gt$; which, by the prop. is to be a maximum. Or, striking out the constant factors, $\frac{1}{m}$, gt, then is $\frac{\text{PWL} - \text{W}^2 l}{\text{P} + \text{W}}$ = a max. Putting this into fluxions, and reducing, we have $\text{P}^2\text{L} - 2\text{PW}l - \text{W}^2 l = 0$, or $\text{W} = \text{P}\sqrt{(\frac{\text{L}}{l} + 1)} - \text{P}$.

Cor. When the inclinations of the planes are equal, L and l are equal, and $\text{W} = \text{P}\sqrt{2} - \text{P} = \text{P} \times (\sqrt{2} - 1) = \cdot 4142\text{P}$; agreeing with the conclusion of the lever of equal arms, or the extreme case of the wheel and axle, i. e. the pulley.

PROPOSITION V.

Given the radius R *of a wheel, and the radius* r, *of its axle, the weight of both,* w, *and the distance of the centre of gyration from the axis of motion,* ϱ; *also a given power* P *acting at the circumference of the wheel; to find the weight* W *raised by a cord folding about the axle, so that its momentum shall be a maximum.*



The force which absolutely impels the point A is P, while W acts in a direction contrary to P, with a force $= \frac{r\text{W}}{\text{R}}$; this therefore subducted from P, leaves $\text{P} - \frac{r\text{W}}{\text{R}} = \frac{\text{RP} - r\text{W}}{\text{R}}$, for the reduced force impelling the point A. And the inertia which resists the communication of motion to the point A will be the same as if the mass $\frac{\varrho^2 w + r^2 \text{W} + \text{R}^2 \text{P}}{\text{R}^2}$ were concentrated in the point A (Mechan. prop. 50). If the former of these be divided by the latter, the quotient $\frac{\text{R}(\text{RP} - r\text{W})}{\varrho^2 w + r^2\text{W} + \text{R}^2\text{P}}$ is the force accelerating A: multiplying

multiplying this by $\frac{r}{R}$, we have $\frac{RrP - r^2W}{\varrho^2 w + r^2W + R^2P}$ for the force which accelerates the weight W in its ascent. Consequently the velocity of W will be $= \frac{RrP - r^2W}{\varrho^2 w + r^2W + R^2P}gt$; which multiplied into W gives $\frac{RrPW - r^2W^2}{\varrho^2 w + r^2W + R^2P}gt$ for the momentum. As this is to be a maximum, its fluxion will $= 0$; whence we shall obtain $W = \frac{\sqrt{(R^4P^2 + 2R^2P\varrho^2 w + \varrho^4 w^2 + PwRr\varrho^2 + P^2R^3r)} - R^2P - \varrho^2 w}{r^2}$.

Cor. 1. When $R = r$, as in the case of the single fixed pulley, then $W = \sqrt{(2P^2R^3 + 2RP\varrho^2 w + \frac{\varrho^4}{R}w^2 + PwR\varrho^2)} - \frac{\varrho^2}{R^2}w - P$.

Cor. 2. When the pulley is a cylinder of uniform matter $\varrho^2 = \frac{1}{2}R^2$, and the express. becomes $W = \sqrt{[R^3(2P^2 + \frac{3}{2}Pw + \frac{1}{4}w^2)]} - \frac{1}{2}w - P$.

Cor. 3. If, in the first general expression for the momentum of W, Q be put $= R^2P + \varrho^2 w$, we shall have $\frac{RrPW - r^2W^2}{Q + r^2W}$ = a maximum. Which, in fluxions and reduced, gives $W = \frac{1}{r^2}\sqrt{Q(Q + RrP)} - \frac{1}{r^2}Q$.

Cor. 4. If the moving force be destitute of inertia, then will $Q = \varrho^2 w$, and W, as in the last corollary.

PROPOSITION VI.

Let a given power P *be applied to the circumference of a wheel, its radius* R, *to raise a weight* W *at its axle, whose radius is* r, *it is required to find the ratio of* R *and* r *when* W *is raised with the greatest momentum, the characters* w *and* ϱ *denoting the same as in the last proposition.*

Here we suppose r to vary in the expression for the momentum of W, $\frac{WRrP - r^2W^2}{\varrho^2 w + r^2W + R^2P}gt$. And we suppose, that by the conditions of any specified instance, we can ascertain what quantity of matter q shall make $r^2q = \varrho^2 w$, which, in fact, may always be done as soon as we can determine ϱ. The expression for the work will then become $\frac{PRrW - r^2W^2}{R^2P + r^2(q + W)}gt$. The fluxion of which being made $= 0$, gives, after a little reduction, $r = \frac{R\sqrt{[P^2W^2 + P^3(q + W)]} - PW}{P(q + W)}$.

Cor. When the inertia of the machine is evanescent, with respect to that of P + W, then is $r = R\sqrt{(1 + \frac{P}{W})} - 1$.

PROPOSITION

PROPOSITION VII.

In any machine whose motion accelerates; the weight will be moved with the greatest velocity, when the velocity of the power is to that of the weight, as $1 + P\sqrt{(1 + \frac{P}{W})}$ *to* 1; *the inertia of the machine being disregarded.*

For any such machine may be considered as reduced to a lever, or to a wheel and axle whose radii are R and r: in which the velocity of the weight $\frac{RrP - r^2W}{R^2P + r^2W}gt$ (prop. I) is to be a maximum, r being considered as variable. Hence then, following the usual rules, we find $PR = r(W + \sqrt{W^2 + PW})$. From which, since the velocities of the power and weight are respectively as R and r, the ratio in the proposition immediately flows.

Cor. When the weight moved is equal to the power, then is $R : r :: 1 + \sqrt{2} : 1 :: 2{\cdot}4142 : 1$ *nearly.*

PROPOSITION VIII.

If in any machine whose motion accelerates, the descent of one weight causes another to ascend, and the descending weight be given, the operation being supposed continually repeated, the effect will be greatest in a given time when the ascending weight is to the descending weight, as 1 *to* 1·618, *in the case of equal heights; and in other cases, when it is to the exact counterpoise in a ratio which is always between* 1 *to* $1\frac{1}{2}$ *and* 1 *to* 2.

Let the space descended be 1, that ascended s; the descending weight 1, the ascending weight $\frac{1}{w}$: then would the equilibrium require $w = s$; and $1 - \frac{s}{w}$ will be the force acting on 1. Now the mass $\frac{1}{w}$, reduced to the point at which the mass 1 acts, will be $= \frac{1}{w}s^2 = \frac{s^2}{w}$; consequently the whole mass moved is equivalent to $1 + \frac{s^2}{w}$, and the relative force is $(1 - \frac{s}{w}) \div (1 + \frac{s^2}{w}) = \frac{w - s}{w + s^2}$. But, the space being given, the time is as the root of the accelerating force inversely, that is, as $\sqrt{\frac{w + s^2}{w - s}}$: and the whole effect in a given time, being directly as the weight raised, and inversely as the time of ascent, will be as $\frac{1}{w}\sqrt{\frac{w - s}{w + s^2}}$; which must be a maximum.

maximum. Consequently its square $\frac{w-s}{w^3+s^2w^2}$ must be a max. likewise. This latter expression, in fluxions and reduced, gives $w = \frac{s}{4}[\sqrt{(s^2 + 10s + 9)} - a + 3]$.

Here if $s = 1$, $w = \frac{1+\sqrt{5}}{2}$: but if s be diminished without limit, $w = \frac{3}{2}s$; if it be augmented without limit, then will $\sqrt{(s^2 + 10s + 9)}$ approach indefinitely near to $s + 5$, and consequently $w = 2s$. Whence the truth of the proposition is manifest.

PROPOSITION IX.

Let φ denote the absolute effort of any moving force, when it has no velocity; and suppose it not capable of any effort when the velocity is W; *let* F *be the effort answering to the velocity* V; *then, if the force be uniform,* F *will be* $= \varphi(1 - \frac{V}{W})^2$.

For it is the difference between the velocities W and V which is efficient, and the action, being constant, will vary as the square of the efficient velocity. Hence we shall have this analogy, $\varphi : F :: (W - 0)^2 : (W - V)^2$: consequently, $F = \varphi(\frac{W-V}{W})^2 = \varphi(1 - \frac{V}{W})^2$.

Though the pressure of an animal is not actually uniform during the whole time of its action, yet it is nearly so: so that in general we may adopt this hypothesis in order to approximate to the true nature of animal action. On which supposition the preceding prop. as well as the remaining one, in this chapter, will apply to animal exertion.

Cor. Retaining the same notation, we have $W = \frac{V\sqrt{\varphi}}{\sqrt{\varphi} - \sqrt{F}}$. This, applied to the motion of animals, gives this theorem: *The utmost velocity with which an animal not impeded can move, is to the velocity with which it moves when impeded by a given resistance, as the square root of its absolute force, to the difference of the square roots of its absolute and efficient forces.*

PROPOSITION X.

To investigate expressions by means of which the maximum effect, in machines whose motion is uniform, may be determined.

I. It follows, from the observations made in art. 1 and the definitions in this chapter, that when a machine, whether simple or compound, is put into motion, the velocities of the impelled

impelled and working points, are inversely as the forces which are in equilibrio, when applied to those points in the direction of their motion. Consequently, if f denote the resistance when reduced to the working point, and v its velocity; while F and V denote the force acting at the impelled point, and its velocity; we shall have $\mathrm{FV} = fv$, or introducing t the time, $\mathrm{FV}t = fvt$. Hence, *in all working machines which have acquired an uniform motion, the performance of the machine is equal to the momentum of impulse.*

II. Let F be the effort of a force on the impelled point of a machine when it moves with the velocity V, the velocity being W when $\mathrm{F} = 0$, and let the relative velocity $\mathrm{W} - \mathrm{V} = u$. Then since (prop. IX) $\mathrm{F} = \phi(\frac{\mathrm{W}-\mathrm{V}}{\mathrm{W}})^2$, the momentum of impulse FV will become $\mathrm{V}\phi(\frac{u}{\mathrm{W}})^2 = \phi \cdot \frac{u^2}{\mathrm{W}^2}(\mathrm{W} - u)$; because $\mathrm{V} = \mathrm{W} - u$. Making this expression for FV a maximum, or, suppressing the constant quantities; and making $u^2(\mathrm{W} - u)$ a max. or its flux. $= 0$, when u is variable, we find $2\mathrm{W} = 3u$, or $u = \frac{2}{3}\mathrm{W}$. Whence $\mathrm{V} = \mathrm{W} - u = \mathrm{W} - \frac{2}{3}\mathrm{W} = \frac{1}{3}\mathrm{W}$.

Consequently, *when the ratio of* V *to* v *is given, by the construction of the machine, and the resistance is susceptible of variation, we must load the machine more or less till the velocity of the impelled point, is one-third of the greatest velocity of the force; then will the work done be a maximum.*

Or, the work done by an animal is greatest, when the velocity with which it moves, is one-third of the greatest velocity with which it is capable of moving when not impeded.

III. Since $\mathrm{F} = \phi\frac{u^2}{\mathrm{W}^2} = \phi(\frac{\frac{4}{9}\mathrm{W}^2}{\mathrm{W}^2}) = \frac{4}{9}\phi$, in the case of the maximum, we have $\mathrm{FV} = \frac{4}{9}\phi\mathrm{V} = \frac{4}{9}\phi \cdot \frac{1}{3}\mathrm{W} = \frac{4}{27}\phi\mathrm{W}$, for the momentum of impulse, or for the work done, when the machine is in its best state. *Consequently, when the resistance is a given quantity, we must make* $\mathrm{V} : v :: 9f : 4\phi$; *and this structure of the machine will give the maximum effect* $= \frac{4}{27}\phi\mathrm{W}$.

IV. If we enquire the greatest effect on the supposition that ϕ only is variable, we must make it infinite in the above expression for the work done, which would then become WF, or $\mathrm{W}\frac{\mathrm{V}}{v}f$, or $\mathrm{W}\frac{\mathrm{V}}{v}ft$, including the time in the formula. Hence we see, *that the sum of the agents employed to move a machine may be infinite, while the effect is finite:* for the variations of ϕ, which are proportional to this sum, do not influence the above expression for the effect.

Scholium.

Scholium.

The propositions now delivered contain the most material principles in the theory of machines. The manner of applying several of them is very obvious: the application of some, being less manifest, may be briefly illustrated, and the chapter concluded with two or three observations.

The last theorem may be applied to the action of men and of horses, with more accuracy than might at first be supposed. Observations have been made on men and horses drawing a lighter along a canal, and working several days together. The force exerted was measured by the curvature and weight of the track-rope, and afterwards by a spring steelyard. The product of the force thus ascertained, into the velocity per hour, was considered as the momentum. In this way the action of *men* was found to be very nearly as $(w-v)^2$: the action of horses loaded so as not to be able to trot was nearly as $(w-v)^{1\cdot 7}$, or as $(w-v)^{\frac{9}{5}}$. Hence the hypothesis we have adopted may in many cases be safely assumed.

According to the best observations, the force of a man at rest is on the average about 70 pounds; and the utmost velocity with which he can walk is about 6 feet per second, taken at a medium. Hence, in our theorems, $\phi = 70$, and $w = 6$. Consequently $F = \frac{4}{9}\phi = 31\frac{1}{9}$lbs, the greatest force a man can exert when in motion: and he will then move at the rate of $\frac{1}{3}w$, or 2 feet per second, or rather less than a mile and a half per hour.

The strength of a horse is generally reckoned about 6 times that of a man; that is, nearly 420 lbs, at a dead pull. His utmost walking velocity is about 10 feet per second. Therefore his maximum action will be $\frac{4}{9}$ of $420 = 186\frac{2}{3}$ lbs, and he will then move at the rate of $\frac{1}{3}$ of 10, or $3\frac{1}{3}$ feet, per second, or nearly $2\frac{1}{3}$ miles per hour. In both these instances we suppose the force to be exerted in drawing a weight along a horizontal plane; or by raising a weight by a cord running over a pulley, which makes its direction horizontal.

2. The theorems just given may serve to show, in what points of view machines ought to be considered, by those who would labour beneficially for their improvement.

The first object of the utility of machines consists in furnishing the means of *giving to the moving force the most commodious direction*; and, when it can be done, of causing its action to be applied immediately to the body to be moved. These can rarely be united: but the former can be accomplished in most instances; of which the use of the simple lever,

lever, pulley, and wheel and axle, furnish many examples. The second object gained by the use of machines, is *an accommodation of the velocity of the work to be performed, to the velocity with which alone a natural power can act.* Thus, whenever the natural power acts with a certain velocity which cannot be changed, and the work must be performed with a greater velocity, a machine is interposed moveable round a fixed support, and the distances of the impelled and working points are taken in the proportion of the two given velocities.

But the essential advantage of machines, that, in fact, which properly appertains to the *theory* of mechanics, consists in augmenting, or rather in modifying, the energy of the moving power, in such manner that it may produce effects of which it would have been otherwise incapable. Thus a man might carry up a flight of steps 20 pieces of stone, each weighing 30 pounds (one by one) in as small a time as he could (with the same labour) raise them all together by a piece of machinery, that would have the velocities of the impelled and working points as 20 to 1; and, in this case, the instrument would furnish no real advantage, except that of saving his steps. But if a large block of 20 times 30, or 600 lbs. weight, were to be raised to the same height, it would far surpass the utmost efforts of the man, without the intervention of some such contrivance.

The same purpose may be illustrated somewhat differently; confining the attention all along to machines whose motion is uniform. The product fv represents, during the unit of time, the effect which results from the motion of the resistance; this motion being produced in any manner whatever. If it be produced by applying the moving force immediately to the resistance, it is necessary not only that the products FV and fv should be equal; but that at the same time $F = f$, and $V = v$: if, therefore, as most frequently happens, f be greater than F, it will be absolutely impossible to put the resistance in motion by applying the moving force immediately to it. Now machines furnish the means of disposing the product FV in such a manner that it may always be equal to fv, however much the factors of FV may differ from the analogous factors in fv; and, consequently, of putting the system in motion, whatever is the excess of f over F.

Or, generally, as M. Prony remarks (Archi. Hydraul. art. 504), machines enable us to dispose the factors of FVt in such a manner, that while that product continues the same, its factors may have to each other any ratio we desire. If, for instance, time be precious, the effect must be produced in a very

short

short time, and yet we should have at command a force capable of little velocity but of great effort, a machine must be found to supply the velocity necessary for the intensity of the force: if, on the contrary, the mechanist has only a weak power at his disposition, but capable of a great velocity, a machine must be adopted that will compensate, by the velocity the agent can communicate to it, for the force wanted: lastly, if the agent is capable neither of great effort, nor of great velocity, a convenient machine may still enable him to accomplish the effect desired, and make the product FVt of force, velocity, and time, as great as is requisite. Thus, to give another example: Suppose that a man, exerting his strength immediately on a mass of 25 lbs, can raise it vertically with a velocity of 4 feet per second; the same man acting on a mass of 1000lbs, cannot give it any vertical motion though he exerts his utmost strength, unless he has recourse to some machine. Now he is capable of producing an effect equal to $25 \times 4 \times t$: the letter t being introduced because, if the labour is continued, the value of t will not be indefinite, but comprised within assignable limits. Thus we have $25 \times 4 \times t = 1000 \times v \times t$; and consequently $v = \frac{1}{10}$ of a foot. This man may therefore with a machine, as a lever, or axis in peritrochio, cause a mass of 1000 lb. to rise $\frac{1}{10}$ of a foot, in the same time that he could raise 25 lbs 4 feet without a machine; or he may raise the greater weight as far as the less, by employing 40 times as much time.

From what has been said on the extent of the effects which may be attained by machines, it will be seen that, so long as a moving force exercises a determinate effort, with a velocity also determinate, or so long as the product of these is constant, the effect of the machine will remain the same: thus, under this point of view, supposing the preponderance of the effort of the moving power, and abstracting from inertia and friction of materials, the convenience of application, &c, all machines are equally perfect. But, from what has been shown, (props. 9, 10) a moving force may, by diminishing its velocity, augment its effort, and reciprocally. There is therefore a certain effort of the moving force, such that its product by the velocity which comports to that effort, is the greatest possible. Admitting the truth of the law assumed in the propositions just referred to, we have, when the effect is a *maximum*, $V = \frac{1}{3}W$, or $F = \frac{4}{9}\varphi$; and these two values obtaining together, their product $\frac{4}{27}\varphi W$ expresses the value of the greatest effect with respect to the unit of time. In practice it will always be advisable to approach as nearly to these values as circumstances will admit; for it cannot be expected

expected that they can always be exactly attained. But a small variation will not be of much consequence: for, by a well-known property of those quantities which admit of a proper maximum and minimum, a value assumed at a moderate distance from either of these extremes will produce no sensible change in the effect.

If the relation of F to V followed any other law than that which we have assumed, we should find from the expression of *that law* values of F, V, &c, different from the preceding. The general method however would be nearly the same.

With respect to practice, the grand object in all cases should be to procure an *uniform motion*, because it is that from which *(cæteris paribus)* the greatest effect always results. Every irregularity in the motion wastes some of the impelling power; and it is the greatest only of the varying velocities which is equal to that which the machine would acquire if it moved uniformly throughout: for, while the motion accelerates, the impelling force is greater than what balances the resistance at that time opposed to it, and the velocity is less than what the machine would acquire if moving uniformly; and when the machine attains its greatest velocity, it attains it because the power is not then acting against the whole resistance. In both these situations therefore, the performance of the machine is less than if the power and resistance were exactly balanced; in which case it would move uniformly (art. 1). Besides this, when the motion of a machine, and particularly a very ponderous one, is irregular, there are continual repetitions of strains and jolts which soon derange and ultimately destroy the whole structure. Every attention should therefore be paid to the removal of all causes of irregularity.

CHAPTER XII.

PRESSURE OF EARTH AND FLUIDS AGAINST WALLS AND FORTIFICATIONS, THEORY OF MAGAZINES, &c.

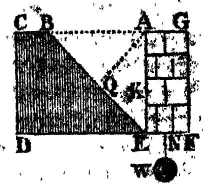
PROBLEM I.

To determine the Pressure of Earth against Walls.

WHEN new-made earth, such as is used in forming ramparts, &c, is not supported by a wall as a facing, or by counterforts and land-ties, &c, but left to the action of its weight and the weather; the particles loosen and separate from each other,

other, and form a sloping surface, nearly regular; which plane surface is called the natural slope of the earth; and is supposed to have always the same inclination or deviation from the perpendicular, in the same kind of soil. In common earth or mould, being a mixture of all sorts thrown together, the natural slope is commonly at about half a right angle, or 45 degrees; but clay and stiff loam stand at a greater angle above the horizon, while sand and light mould will only stand at a much less angle. The engineer or builder must therefore adapt his calculations accordingly.

Now, we have already given, at prop 45 Statics in vol. 2, 6th edition, the general theory and determination of the force with which the triangle of earth (which would slip down if not supported) presses against the wall on the most unexceptionable principles, acting perpendicularly against AE at K, or $\frac{1}{3}$ of the altitude AE above the foundation at E; the expression for which force was there found to be $\frac{AE^3 \cdot AB^2}{6BE^2}m$; where m denotes the specific gravity of the earth of the triangle ABE.—It may be remarked that this was deduced from using the area only of the profile, or transverse triangular section ABE, instead of the prismatic solid of any given length, having that triangle for its base. And the same thing is done in determining the power of the wall to support the earth, viz, using only its profile or transverse section in the same plane or direction as the triangle ABE. This it is evident will produce the same result as the solids themselves, since, being both of the same given length, these have the same ratio as their transverse sections.



In addition to this determination, we may here further observe, that this pressure ought to be diminished in proportion to the cohesion of the matter in sliding down the inclined plane BE. Now it has been found by experiments, that a body requires about one-third of its weight to move it along a plane surface. The above expression must therefore be reduced in the ratio of 3 to 2; by which means it becomes $\frac{AE^3 \cdot AB^2}{9BE^2}m$ for the true practical efficacious pressure of the earth against the wall.

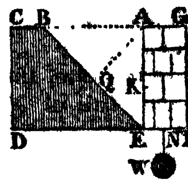
Since $\frac{AB}{BE}$, which occurs in this expression of the force of the earth, is equal to the sine of the $\angle$ AEB to the radius 1, put the sine of that $\angle$ E $= e$; also put $a =$ AE the altitude of the triangle; then the above expression of the force, viz,

 $AE^3 \cdot AB^2$

$\frac{AE^3 \cdot AB^2}{9BE^2}m$, becomes $\frac{1}{9}a^3e^2m$, for the perpendicular pressure of the earth against the wall. And if that angle be 45°, as is usually the case in common earth, then is $e^2 = \frac{1}{2}$, and the pressure becomes $\frac{1}{18}a^3m$.

PROBLEM II.

To determine the Thickness of Wall to support the Earth.



In the first place suppose the section of the wall to be a rectangle, or equally thick at top and bottom, and of the same height as the rampart of earth, like AEFG in the annexed figure. Conceive the weight W, proportional to the area GE, to be appended to the base directly below the centre of gravity of the figure. Now the pressure of the earth determined in the first problem, being in a direction parallel to AG, to cause the wall to overset and turn back about the point F, the effort of the wall to oppose that effect, will be the weight W drawn into FN the length of the lever by which it acts, that is W × FN, or AEFG × FN in general, whatever be the figure of the wall.

But now in case of the rectangular figure, the area GE = AE × EF $= ax$, putting a = AE the altitude as before, and x = EF the required thickness; also in this case FN $= \frac{1}{2}$EF $= \frac{1}{2}x$, the centre of gravity being in the middle of the rectangle. Hence then $ax \times \frac{1}{2}x = \frac{1}{2}ax^2$, or rather $\frac{1}{2}ax^2n$ is the effort of the wall to prevent its being overturned, n denoting the specific gravity of the wall.

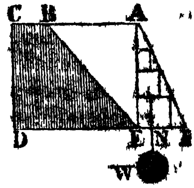
Now to make this effort a due balance to the pressure of the earth, we put the two opposing forces equal, that is $\frac{1}{2}ax^2n = \frac{1}{9}a^3e^2m$, or $\frac{1}{2}x^2n = \frac{1}{9}a^2e^2m$, an equation which gives $x = \frac{1}{3}ae\sqrt{\frac{2m}{n}}$, for the requisite thickness of the wall, just to sustain it in equilibrio.

Corol. 1. The factor ae, in this expression, is = the line AQ drawn perp. to the slope of earth BE: theref. the breadth x becomes $= \frac{1}{3}$AQ$\sqrt{\frac{2m}{n}}$, which conseq. is directly proportional to the perp. AQ.—When the angle at E is = 45°, or half a right angle, as is commonly the case, its sine e is $= \sqrt{\frac{1}{2}}$, and the breadth of the wall $x = \frac{1}{3}a\sqrt{\frac{m}{n}}$. Further, when the wall is of brick, its specific gravity is nearly the same as the

the earth, or $m = n$, and then its thickness $x = \frac{1}{3}a$, or one-third of its height.—But when the wall is of stone, of the specific gravity $2\frac{1}{2}$, that of earth being nearly 2, that is, $m = 2$, and $n = 2\frac{1}{2}$; then $\sqrt{\frac{m}{n}} = \sqrt{\frac{4}{5}} = \cdot 895$, $\frac{1}{3}$ of which is $\cdot 298$, and the breadth $x = \cdot 298a = \frac{3}{10}a$ nearly. That is, the thickness of the stone wall must be $\frac{3}{10}$ of its height.

PROBLEM III.

To determine the Thickness of the Wall at the Bottom, when its Section is a Triangle, or coming to an Edge at Top.



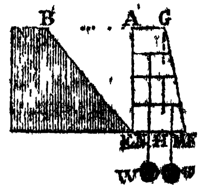
In this case, the area of the wall AEF is only half of what it was before, or only $\frac{1}{2}$AE × EF $= \frac{1}{2}ax$, and the weight W $= \frac{1}{2}axn$. But now, the centre of gravity is at only $\frac{1}{3}$ of FE from the line AE, or FN $= \frac{2}{3}$FE $= \frac{2}{3}x$. Consequently FN × W $= \frac{2}{3}x \times \frac{1}{2}axn = \frac{1}{3}ax^2n$. This, as before, being put = the pressure of the earth, gives the equation $\frac{1}{3}ax^2n = \frac{1}{9}a^3e^2m$, or $x^2n = \frac{1}{3}a^2e^2m$, and the root x, or thickness EF $= ae\sqrt{\frac{m}{3n}} = a\sqrt{\frac{m}{6n}}$ for the slope of 45°.

Now when the wall is of brick, or $m = n$ nearly, this becomes $x = a\sqrt{\frac{1}{6}} = \cdot 408a = \frac{2}{5}a$, or $\frac{4}{10}$ of the height nearly.

But when the wall is of stone, or m to n as 2 to $2\frac{1}{2}$, then $\sqrt{\frac{m}{n}} = \sqrt{\frac{4}{5}}$, and the thickness x or $a\sqrt{\frac{m}{6n}} = a\sqrt{\frac{2}{15}} = \cdot 365a = \frac{3}{8}a$ nearly, or nearly $\frac{3}{8}$ of the height.

PROBLEM IV.

To determine the Thickness of the Wall at the Top, when the Face is not Perpendicular, but Inclined as the Front of a Fortification Wall usually is.



Here GF represents the outer face of a fort, AEFG the profile of the wall, having AG the thickness at top, and EF that at the bottom. Draw GH perp. to EF; and conceive the two weights W, w, to be suspended from the centres of gravity of the rectangle AH and the triangle GHF, and to be proportional to their areas respectively. Then the two momenta of the weights W, w, acting by the levers FN, FM, must be made equal to the pressure of the earth in the direction perp. to AE.

 Now

Now put the required thickness AG or EH $= x$, and the altitude AE or GH $= a$ as before. And because in such cases the slope of the wall is usually made equal to $\frac{1}{5}$ of its altitude, that is FH $= \frac{1}{5}$AE or $\frac{1}{5}a$, the lever FM will be $\frac{2}{3}$ of $\frac{1}{5}a = \frac{2}{15}a$, and the lever FN $=$ FH $+ \frac{1}{2}$EH $= \frac{1}{5}a + \frac{1}{2}x$. But the area of GHF $=$ GH $\times \frac{1}{2}$HF $= a \times \frac{1}{10}a = \frac{1}{10}a^2 = w$, and the area AH $=$ AE $\times$ AG $= ax = W$; these two drawn into the respective levers FM, FN, give the two momenta, $\frac{2}{15}aw = \frac{2}{15}a \times \frac{1}{10}a^2 = \frac{1}{75}a^3$, and $(\frac{1}{5}a + \frac{1}{2}x) \times ax = \frac{1}{5}a^2x + \frac{1}{2}ax^2$; theref. the sum of the two, $(\frac{1}{2}ax^2 + \frac{1}{5}a^2x + \frac{1}{75}a^3)n$ must be $= \frac{1}{18}a^3m$, or dividing by $\frac{1}{2}an$, $x^2 + \frac{2}{5}ax + \frac{2}{75}a^2 = \frac{1}{9}a^2 \times \frac{m}{n}$; now adding $\frac{1}{75}a^2$ to both sides to complete the square, the equation becomes $x^2 + \frac{2}{5}ax + \frac{1}{25}a^2 = \frac{1}{9}a^2 \cdot \frac{m}{n} + \frac{1}{75}a^2$ the root of which is $x + \frac{1}{5}a = a\sqrt{(\frac{1}{25} + \frac{m}{9n})}$, and hence $x = a\sqrt{(\frac{1}{25} + \frac{m}{9n})} - \frac{1}{5}a$. And the base EF $= a\sqrt{(\frac{1}{25} + \frac{m}{9n})}$.

Now, for a brick wall, $m = n$ nearly, and then the breadth $x = a\sqrt{(\frac{1}{25} + \frac{1}{9})} - \frac{1}{5}a = \frac{1}{15}a\sqrt{34} - \frac{1}{5}a = \cdot 189a$, or almost $\frac{1}{5}a$ in brick walls.—But in stone walls, $\frac{m}{n} = \frac{4}{5}$, and $x = a\sqrt{(\frac{1}{25} + \frac{4}{45})} - \frac{1}{5}a = \frac{1}{15}a\sqrt{29} - \frac{1}{5}a = \cdot 159a = \frac{4}{25}a$ nearly, for the thickness AG at top, in stone walls.

In the same manner we may proceed when the slope is supposed to be any other part of the altitude, instead of $\frac{1}{5}$ as used above. Or a general solution might be given, by assuming the thickness $= \frac{1}{c}$ part of the altitude.

REMARK.

Thus then we have given all the calculations that may be necessary in determining the thickness of a wall, proper to support the rampart or body of earth, in any work. If it should be objected, that our determination gives only such a thickness of wall, as makes it an exact mechanical balance to the pressure or push of the earth, instead of giving the former a decided preponderance over the latter, as a security against any failure or accidents. To this we answer, that what has been done is sufficient to insure stability, for the following reasons and circumstances. First, it is usual to build several counterforts of masonry, behind and against the wall, at certain distances or intervals from one another; which contribute very much to strengthen the wall, and to resist the pressure of the rampart. 2dly. We have omitted to include the effect of the parapet raised above the wall; which must add somewhat, by its weight, to the force or resistance of the wall.

wall. It is true we could have brought these two auxiliaries to exact calculation, as easily as we have done for the wall itself: but we have thought it as well to leave these two appendages, thrown in as indeterminate additions, above the exact balance of the wall as before determined, to give it an assured stability. Besides these advantages in the wall itself, certain contrivances are also usually employed to diminish the pressure of the earth against it: such as land-ties and branches, laid in the earth, to diminish its force and push against the wall. For all these reasons then, we think the practice of making the wall of the thickness as assigned by our theory, may be safely depended on, and profitably adopted; as the additional circumstances, just mentioned, will sufficiently insure stability; and its expense will be less than is incurred by any former theory.

PROBLEM V.

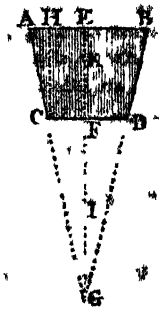
To determine the Quantity of Pressure sustained by a Dam or Sluice, made to pen up a Body of Water.

By art. 313 Hydrostatics, vol. 2, 6th edit. the pressure of a fluid against any upright surface, as the gate of a sluice or canal, is equal to half the weight of a column of the fluid, whose base is equal to the surface pressed, and its altitude the same as that of the surface. Or, by art. 314 of the same, the pressure is equal to the weight of a column of the fluid, whose base is equal to the surface pressed, and its altitude equal to the depth of the centre of gravity below the top or surface of the water; which comes to the same thing as the former article, when the surface pressed is a rectangle, because its centre of gravity is at half the depth.

Ex. 1. Suppose the dam or sluice be a rectangle, whose length, or breadth of the canal, is 20 feet, and the depth of water 6 feet. Here $20 \times 6 = 120$ feet, is the area of the surface pressed; and the depth of the centre of gravity being 3 feet, viz, at the middle of the rectangle; therefore $120 \times 3 = 360$ cubic feet is the content of the column of water. But each cubic foot of water weighs 1000 ounces, or $62\frac{1}{2}$ pounds; therefore $360 \times 1000 = 360000$ ounces, or 22500 pounds, or 10 tons and 100 lb, is the weight of the column of water, or the quantity of pressure on the gate or dam.

Ex. 2. Suppose the breadth of a canal at the top, or surface of the water, to be 24 feet, but at the bottom only 16 feet, the depth of water being 6 feet, as in the last example: required the pressure on a gate which, standing across the canal, dams the water up?

Here



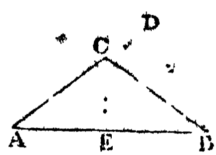
Here the gate is in form of a trapezoid, having the two parallel sides AB, CD, viz, AB $= 24$, and CD $= 16$, and depth 6 feet. Now, by mensuration, problem 3 vol. 2, $\frac{1}{2}(AB + CD) \times 6 = 20 \times 6 = 120$ the area of the sluice, the same as before in the 1st example: but the centre of gravity cannot be so low down as before, because the figure is wider above and narrower below, the whole depth being the same.

Now, to determine the centre of gravity K of the trapezoid AD, produce the two sides AC, BD, till they meet in G; also draw GKE and CH perp. to AB: then AH : CH :: AE : GE, that is, $4 : 6 :: 12 : 18 =$ GE; and EF being $= 6$, theref. FG $= 12$. Now, by Statics art. 229 vol. 2, EF $= 6 = \frac{1}{3}$EG gives F the centre of gravity of the triangle ABG, and FI $= 4 = \frac{1}{3}$FG gives I the centre of gravity of the triangle CDG. Then assuming K to denote the centre of AD, it will be, by art. 212 vol. 2, as the trap. AD : $\triangle$ CDG :: IF : FK, or $\triangle$ ABG $-$ $\triangle$ CDG : $\triangle$ CDG :: IF : FK, or by theor. 88 Geom. $GE^2 - GF^2 : GF^2 :: IF : FK$, that is $18^2 - 12^2$ to 12^2 or $3^2 - 2^2$ to 2^2 or $5 : 4 ::$ IF $= 4 : \frac{16}{5} = 3\frac{1}{5} =$ FK; and hence EK $= 6 - 3\frac{1}{5} = 2\frac{4}{5} = \frac{14}{5}$ is the distance of the centre K below the surface of the water. This drawn into 120 the area of the dam-gate, gives 336 cubic feet of water = the pressure, = 336000 ounces = 21000 pounds = 9 tons 80 lb, the quantity of pressure against the gate, as required, being a 15th part less than in the first case.

Ex. 3. Find the quantity of pressure against a dam or sluice, across a canal, which is 20 feet wide at top, 14 at bottom, and 8 feet depth of water?

PROBLEM VI.

To determine the Strongest Angle of Position of a Pair of Gates for the Lock on a Canal or River.



Let AC, BC be the two gates, meeting in the angle C, projecting out against the pressure of the water, AB being the breadth of the canal or river. Now the pressure of the water on a gate AC, is as the quantity, or as the extent or length of it, AC. And the mechanical effect of that pressure, is as the length of lever to the middle of AC, or as AC itself. On both these accounts then the pressure is as

AC^2.

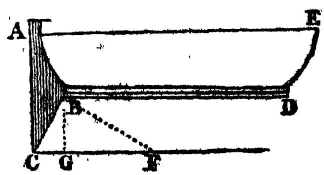
AC². Therefore the resistance or the strength of the gate must be as the reciprocal of this AC².

Now produce AC to meet BD, perp. to it, in D; and draw CE to bisect AB perpendicularly in E; then, by similar triangles, as AC : AE :: AB : AD; where, AE and AB being given lengths, AD is reciprocally as AC, or AD² reciprocally as AC²; that is, AD² is as the resistance of the gate AC. But the resistance of AC is increased by the pressure of the other gate in the direction BC. Now the force in BC is resolved into the two BD, DC; the latter of which, DC, being parallel to AC, has no effect upon it; but the former, BD, acts perpendicularly on it. Therefore the whole effective strength or resistance of the gate is as the product AD² × BD.

If now there be put AB $= a$, and BD $= x$, then AD² $=$ AB² $-$ BD² $= a^2 - x^2$; conseq. AD² × BD $= (a^2 - x^2) \times x = a^2x - x^3$ for the resistance of either gate. And, if we would have this to be the greatest, or the resistance a maximum, its fluxion must vanish, or be equal to nothing: that is, $a^2\dot{x} - 3x^2\dot{x} = 0$; hence $a^2 = 3x^2$, and $x = a\sqrt{\frac{1}{3}} = \frac{1}{3}a\sqrt{3} = \cdot57735a$, the natural sine of 35° 16′: that is, the strongest position for the lock gates, is when they make the angle A or B = 35° 16′, or the complemental angle ACE or BCE = 54° 44′, or the whole salient angle ACB = 109° 28′.

Scholium.

Allied to this problem, are several other cases in mechanics: such as, the action of the water on the rudder of a ship, in sailing, to turn the ship about, to alter her course; and the action of the wind on a ship's sails, to impel her forward; also the action of water on the wheels of water-mills, and of the air on the sails of wind-mills, to cause them to turn round.



Thus, for instance, let ABC be the rudder of a ship ABDE, sailing in the direction BD, the rudder placed in the oblique position BC, and consequently striking the water in the direction CF, parallel to BD. Draw BF perp. to BC, and BG perp. to CF. Then the sine of the angle of incidence, of the direction of the stroke of the rudder against the water, will be BF, to the radius CF; therefore the force of the water against the rudder will be as BF², by art. 3 pa. 366 vol. 2. But the force BF resolves into the two BG, GF, of which the latter is parallel to the ship's motion, and therefore has no effect

effect to change it; but the former BC, being perp. to the ship's motion, is the only part of the force to turn the ship about and change her course. But BF : BG :: CF : CB, therefore CF : CB :: BF^2 : $\frac{BC \cdot BF^2}{CF}$ the force upon the rudder to turn the ship about.

Now put $a = CF$, $x = BC$; then $BF^2 = a^2 - x^2$, and the force $\frac{BC \cdot BF^2}{CF} = \frac{x(a^2 - x^2)}{a} = \frac{a^2x - x^3}{a}$; and, to have this a maximum, its flux. must be made to vanish, that is, $a^2\dot{x} - 3x^2\dot{x} = 0$; and hence $x = a\sqrt{\frac{1}{3}}$ = BC = the natural sine of 35° 16′ = angle F; therefore the complemental angle C = 54° 44′ as before, for the obliquity of the rudder, when it is most efficacious.

The case will be also the same with respect to the wind acting on the sails of a wind-mill, or of a ship, viz, that the sails must be set so as to make an angle of 54° 44′ with the direction of the wind; at least at the beginning of the motion, or nearly so when the velocity of the sail is but small in comparison with that of the wind; but when the former is pretty considerable in respect of the latter, then the angle ought to be proportionally greater, to have the best effect, as shown in Maclaurin's Fluxions, pa. 734, &c.

A consideration somewhat related to the same also, is the greatest effect produced on a mill-wheel, by a stream of water striking upon its sails or float-boards. The proper way in this case seems to be, to consider the whole of the water as acting on the wheel, but striking it only with the relative velocity, or the velocity with which the water overtakes and strikes upon the wheel in motion, or the difference between the velocities of the wheel and the stream. This then is the power or force of the water; which multiplied by the velocity of the wheel, the product of the two, viz, of the relative velocity and the absolute velocity of the wheel, that is $(V - v)v = Vv - v^2$, will be the effect of the wheel; where V denotes the given velocity of the water, and v the required velocity of the wheel. Now, to make the effect $Vv - v^2$ a maximum, or the greatest, its fluxion must vanish, that is $V\dot{v} - 2v\dot{v} = 0$; hence $v = \frac{1}{2}V$; or the velocity of the wheel will be equal to half the velocity of the stream, when the effect is the greatest; and this agrees best with experiments.

A former way of resolving this problem was, to consider the water as striking the wheel with a force as the square of the relative velocity, and this multiplied by the velocity of the wheel, to give the effect; that is, $(V - v)^2v$ = the effect. Now the flux. of this product is $(V - v)^2\dot{v} - (V - v) \times 2v\dot{v} = 0$;

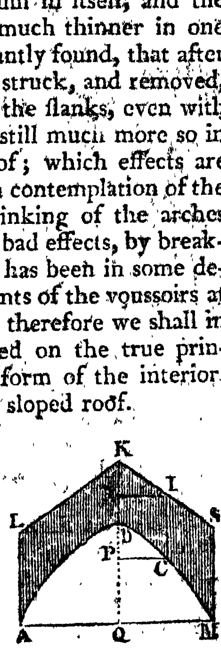
hence

hence $V - v = 2v$, or $V = 3v$, and $v = \frac{1}{3}V$, or the velocity of the wheel equal only to $\frac{1}{3}$ of the velocity of the water.

PROBLEM VII.

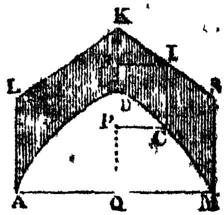
To determine the Form and Dimensions of Gunpowder Magazines.

In the practice of engineering, with respect to the erection of powder magazines, the exterior shape is usually made like the roof of a house, having two sloping sides, forming two inclined planes, to throw off the rain, and meeting in an angle or ridge at the top; while the interior represents a vault, more or less extended, as the occasion may require; and the shape, or transverse section, in the form of some arch, both for strength and commodious room, for placing the powder barrels. It has been usual to make this interior curve a semicircle. But, against this shape, for such a purpose, I must enter my decided protest; as it is an arch the farthest of any from being in equilibrium in itself, and the weakest of any, by being unavoidably much thinner in one part than in others. Besides, it is constantly found, that after the centering of semicircular arches is struck, and removed, they settle at the crown, and rise up at the flanks, even with a straight horizontal form at top, and still much more so in powder magazines with a sloping roof; which effects are exactly what might be expected from a contemplation of the true theory of arches. Now this shrinking of the arches must be attended with other additional bad effects, by breaking the texture of the cement, after it has been in some degree dried, and also by opening the joints of the voussoirs at one end. Instead of the circular arch therefore we shall in this place give an investigation, founded on the true principles of equilibrium, of the only just form of the interior, which is properly adapted to the usual sloped roof.



For this purpose, put a = DK the thickness of the arch at the top, x = any absciss DP of the required arch ADCM; u = KR the corresponding absciss of the given exterior line KI, and y = PC = RI their equal ordinates. Then by the principles of arches, in my tracts on that subject, it is found that CI or $w = a + x - u = Q \times \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3}$, or $= Q \times \frac{\ddot{x}}{\dot{y}^2}$, supposing $\dot{y}$ a constant quantity, and where Q is some certain quantity to be determined hereafter. But KR or u is $= ty$, if t be put to denote the

the tangent of the given angle of elevation KIR, to radius 1; and then the equation is $w = a + x - ty = \frac{Q\ddot{x}}{\dot{y}^2}$.



Now, the fluxion of the equation $w = a + x - ty$, is $\dot{w} = \dot{x} - t\dot{y}$, and the 2d fluxion is $\ddot{w} = \ddot{x}$; therefore the foregoing general equation becomes $w = \frac{Q\ddot{w}}{\dot{y}^2}$; and hence $w\dot{w} = \frac{Q\dot{w}\ddot{w}}{\dot{y}^2}$, the fluent of which gives $w^2 = \frac{Q\dot{w}^2}{\dot{y}^2}$: but at D the value of w is $= a$, and $\dot{w} = 0$, the curve at D being parallel to KI; therefore the correct fluent is $w^2 - a^2 = \frac{Q\dot{w}^2}{\dot{y}^2}$. Hence then $\dot{y}^2 = \frac{Q\dot{w}^2}{w^2 - a^2}$, or $\dot{y} = \frac{\dot{w}\sqrt{Q}}{\sqrt{(w^2 - a^2)}}$; the correct fluent of which gives $y = \sqrt{Q} \times$ hyp. log. of $\frac{w + \sqrt{(w^2 - a^2)}}{a}$.

Now, to determine the value of Q, we are to consider that when the vertical line CI is in the position AL or MN, then $w =$ CI becomes $=$ AL or MN $=$ the given quantity c suppose, and $y =$ AQ or QM $= b$ suppose, in which position the last equation becomes $b = \sqrt{Q} \times$ hyp. log. $\frac{c + \sqrt{(c^2 - a^2)}}{a}$; and hence it is found that the value of the constant quantity $\sqrt{Q}$, is $\frac{b}{\text{h. l } \frac{c + \sqrt{(c^2 - a^2)}}{a}}$; which being substituted for it, in the above general value of y, that value becomes

$$y = b \times \frac{\log \text{ of } \frac{w + \sqrt{(w^2 - a^2)}}{a}}{\log. \text{ of } \frac{c + \sqrt{(c^2 - a^2)}}{a}} = b \times \frac{\log. \text{ of } w + \sqrt{(w^2 - a^2)} - \log. a}{\log \text{ of } c + \sqrt{(c^2 - a^2)} - \log a};$$

from which equation the value of the ordinate PC may always be found, to every given value of the vertical CI.

But if, on the other hand, PC be given, to find CI, which will be the more convenient way, it may be found in the following manner: Put A $=$ log. of a, and C $= \frac{1}{b} \times$ log. of $\frac{c + \sqrt{(c^2 - a^2)}}{a}$; then the above equation gives Cy + A $=$ log. of $w + \sqrt{(w^2 - a^2)}$; again, put $n =$ the number whose log. is Cy + A; then $n = w + \sqrt{(w^2 - a^2)}$; and hence $w = \frac{a^2 + n^2}{2n} =$ CI.

Now, for an example in numbers, in a real case of this nature,

nature, let the foregoing figure represent a transverse vertical section of a magazine arch balanced in all its parts, in which the span or width AM is 20 feet, the pitch or height DQ is 10 feet, thickness at the crown DK = 7 feet, and the angle of the ridge LKS 112° 37′, or the half of it LKD = 56° 18′$\frac{1}{2}$, the complement of which, or the elevation KIR, is 33° 41′$\frac{1}{2}$, the tangent of which is = $\frac{2}{3}$, which will therefore be the value of t in the foregoing investigation. The values of the other letters will be as follows, viz, DK = a = 7; AQ = b = 10; DQ = h = 10; AL = c = $10\frac{1}{3} = \frac{31}{3}$; A = log. of 7 = ·8450980; C = $\frac{1}{b}$ × log. of $\frac{c + \sqrt{(c^2 - a^2)}}{a}$ = $\frac{1}{10}$ log. of $\frac{31 + \sqrt{520}}{21}$ = $\frac{1}{10}$ log. of 2·56207 = ·0408591; Cy + A = ·0408591y + ·8450980 = log. of n. From the general equation then, viz, CI = $w = \frac{a^2 + n^2}{2n} = \frac{a^2}{2n} + \frac{1}{2}n$, by assuming y successively equal to 1, 2, 3, 4, &c, thence finding the corresponding values of Cy + A or ·0408591y + ·8450980, and to these, as common logs. taking out the corresponding natural numbers, which will be the values of n; then the above theorem will give the several values of w or CI, as they are here arranged in the annexed table, from which the figure of the curve is to be constructed, by thus finding so many points in it.

Val. of y or CP.	Val. of w or CI.
1	7·0309
2	7·1243
3	7·2806
4	7·5015
5	7·7888
6	8·1452
7	8·5737
8	9·0781
9	9·6628
10	10·3333

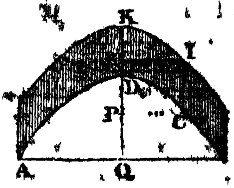
Otherwise. Instead of making n the number of the log. Cy + A, if we put m = the natural number of the log. Cy only; then $m = \frac{w + \sqrt{(w^2 - a^2)}}{a}$, and $am - w = \sqrt{(w^2 - a^2)}$, or by squaring, &c, $a^2m^2 - 2amw + w^2 = w^2 - a^2$, and hence $w = \frac{m^2 + 1}{2m} \times a$: to which the numbers being applied, the very same conclusions result as in the foregoing calculation and table.

PROBLEM VIII.

To construct Powder Magazines with a Parabolical Arch.

It has been shown, in my tract on the Principles of Arches of Bridges, that a parabolic arch is an arch of equilibration, when its extrados, or form of its exterior covering, is the very same parabola as the lower or inside curve. Hence then a parabolic arch, both for the inside and outer form, will be very

very proper for the structure of a powder magazine; For, the inside parabolic shape will be very convenient as to room for stowage: 2dly, the exterior parabola, every where parallel to the inner one, will be proper enough to carry off the rain water: 3dly, the structure will be in perfect equilibrium: and 4thly the parabolic curve is easily constructed, and the fabric erected.



Put, as before, $a =$ KD, $h =$ DQ, $b =$ AQ, $x =$ DP, and $y =$ PC or RI. Then, by the nature of the parabola ADC, $b^2 : y^2 :: h : x = \frac{hy^2}{b^2}$; hence $\dot{x} = \frac{2hy\dot{y}}{bb}$, and $\ddot{x} = \frac{2h\dot{y}^2}{bb}$, by making $\dot{y}$ constant. Then CI $= \frac{\ddot{x}}{\dot{y}^2} \times$ Q is $= \frac{2hQ}{bb} =$ a constant quantity $= a$, what it is at the vertex; that is, CI is everywhere equal to KD.

Consequently KR is $=$ DP, and since RI is $=$ PC, it is evident that KI is the same parabolic curve with DC, and may be placed any height above it, always producing an arch of equilibration, and very commodious for powder magazines.

CHAPTER XIII.

THEORY AND PRACTICE OF GUNNERY.

In the 2d vol. of this course have been given several particulars relating to this subject. Thus, in props. 19, 20, 21, 22, p. 151 &c, is given all that relates to the parabolic theory of projectiles, that is, the mathematical principles which would take place and regulate such projects, if they were not impeded and disturbed in their motions by the air in which they move. But, from the enormous resistance of that medium, it happens, that many military projectiles, especially the smaller balls discharged with the higher velocities, do not range so far as a 20th part of what they would naturally do in empty space! That theory therefore can only be useful in some few cases, such as in the slower kind of motions, not above the velocities of 2, 3, or 400 feet per second, when the path of the projectile differs but little perhaps from the curve of a parabola.

Again,

Again, at pa. 160 &c. are given several other practical rules and calculations, depending partly on the foregoing parabolic theory, and partly on the results of certain experiments performed with cannon balls.

Again, in prop. 58, pa. 219, are delivered the theory and calculations of a beautiful military experiment, invented by Mr. Robins, for determining the true degree of velocity with which balls are projected from guns, with any charges of powder. The idea of this experiment, is simply, that the ball is discharged into a very large but moveable block of wood, whose small velocity, in consequence of that blow, can be easily observed and accurately measured. Then, from this small velocity, thus obtained, the great one of the ball is immediately derived by this simple proportion, viz, as the weight of the ball, is to the sum of the weights of the ball and the block, so is the observed velocity of the last, to a 4th proportional, which is the velocity of the ball sought.—It is evident that this simple mode of experiment will be the source of numerous useful principles, as results derived from the experiments thus made, with all lengths and sizes of guns, with all kinds and sizes of balls and other shot, and with all the various sorts and quantities of gunpowder; in short, the experiment will supply answers to all enquiries in projectiles, excepting the extent of their ranges; for it will even determine the resistance of the air, by causing the ball to strike the block of wood at different distances from the gun, thus showing the velocity lost by passing through those different spaces of air; all which circumstances are fully shown in vols. 2 and 3 of my Tracts just published.

Lastly, in prob. 17 on Forces, near the end of volume 2, some results of the same kind of experiment are successfully applied to determine the curious circumstances of the first force or elasticity of the air resulting from fired gunpowder, and the velocity with which it expands itself. These are circumstances which have never before been determined with any precision. Mr. Robins, and other authors, it may be said, have only guessed at, rather than determined them. That ingenious philosopher, by a simple experiment, truly showed that by the firing of a parcel of gunpowder, a quantity of elastic air was disengaged, which, when confined in the space only occupied by the powder before it was fired, was found to be near 250 times stronger than the weight or elasticity of the common atmospheric air. He then heated the same parcel of air to the degree of red hot iron, and found it in that temperature to be about 4 times as strong as before; whence he inferred, that the first strength of the inflamed

flamed fluid, must be nearly 1000 times the pressure of the atmosphere. But this was merely guessing at the degree of heat in the inflamed fluid, and consequently of its first strength, both which in fact are found to be much greater. It is true, that this assumed degree of strength accorded pretty well with that author's experiments; but this seeming agreement, it may easily be shown, could only be owing to the inaccuracy of his own further experiments; and, in fact, with far better opportunities than fell to the lot of Mr. Robins, we have shown that inflamed gunpowder is about double the strength that he has assigned to it, and that it expands itself with the velocity of about 5000 feet per second.

Fully sensible of the importance of experiments of this kind, first practised by Mr. Robins with musket balls only, my endeavours for many years were directed to the prosecution of the same, on a larger scale, with cannon balls; and having had the honour to be called on to give my assistance at several courses of such experiments, carried on at Woolwich by the ingenious officers of the Royal Artillery there, under the auspices of the Masters General of the Ordnance, I have assiduously attended them for many years. The first of these courses was performed in the year 1775, being 2 years after my establishment in the Royal Academy at that place: and in the Philos. Trans. for the year 1778 I gave an account of these experiments, with deductions, in a memoir, which was honoured with the Royal Society's gold medal of that year. In conclusion, from the whole, the following important deductions were fairly drawn and stated, viz.

1*st*, It is made evident by these experiments, that gunpowder fires almost instantaneously. 2*dly*, The velocities communicated to shot of the same weight, with different charges of powder, are nearly as the square roots of those charges. 3*dly*, And when shot of different weights are fired with the same charge of powder, the velocities communicated to them, are nearly in the inverse ratio of the square roots of their weights. 4*thly*, So that, in general, shot which are of different weights, and impelled by the firing of different charges of powder, acquire velocities which are directly as the square roots of the charges of powder, and inversely as the square roots of the weights of the shot. 5*thly*, It would therefore be a great improvement in artillery, occasionally to make use of shot of a long shape, or of heavier matter, as lead; for thus the momentum of a shot, when discharged with the same charge of powder, would be increased in the ratio of the square root of the weight of the shot; which would both augment proportionally the force of the blow with which

which it would strike, and the extent of the range to which it would go. *6thly*, It would also be an improvement, to diminish the windage; since by this means, one third or more of the quantity of powder might be saved. *7thly*, When the improvements mentioned in the last two articles are considered as both taking place, it appears that about half the quantity of powder might be saved. But, important as this saving may be, it appears to be still exceeded by that of the guns: for thus a small gun may be made to have the effect and execution of another of two or three times its size in the present way, by discharging a long shot of 2 or 3 times the weight of its usual ball, or round shot; and thus a small ship might employ shot as heavy as those of the largest now used.

Finally, as these experiments prove the regulations with respect to the weight of powder and shot, when discharged from the same piece of ordnance; so, by making similar experiments with a gun varied in its length, by cutting off from it a certain part, before each set of trials, the effects and general rules for the different lengths of guns, may be with certainty determined by them. In short, the principles on which these experiments were made, are so fruitful in consequences, that, in conjunction with the effects of the resistance of the medium, they appear to be sufficient for answering all the inquiries of the speculative philosopher, as well as those of the practical artillerist.

Such then was the summary conclusion from the first set of experiments with cannon balls, in the year 1775, and such were the probable advantages to be derived from them. I am not aware however that any alterations were adopted from them by authority in the public service: unless we are to except the instance of carronades, a species of ordnance that was afterwards invented, and in some degree adopted in the public service; for, in this instance, the proprietors of those pieces, by availing themselves of the circumstances of large balls, and very small windage, have, with small charges of powder, and at little expense, been enabled to produce very considerable and useful effects with those light pieces.

The 2d set of these experiments extended through most part of the summer seasons of the years 1783, 1784, 1785, and some in 1786. The objects of this course were numerous and various: but the principal articles as follow: 1. The velocities with which balls are projected by equal charges of powder, from pieces of equal weight and calibre, but of different lengths. 2. The velocities with different charges of powder, the weight and length of the guns being equal. 3. The greatest velocities due to the different lengths of guns, to

to be ascertained by successively increasing the charge, till the bore should be filled, or till the velocity should decrease again. 4. The effect of varying the weight of the piece; every thing else being the same. 5. The penetrations of balls into blocks of wood. 6. The ranges and times of flight of balls; to compare them with their first velocities, for ascertaining the resistance of the medium. 7. The effect of wads; of different degrees of ramming, or compressing the charge; of different degrees of windage; of different positions of the vent; of chambers and trunnions, and every other circumstance necessary to be known for the improvement of artillery.

An ample account is given of these experiments, and the results deduced from them in my volume of Tracts published in 1786; some few circumstances only of which can be noted here. In this course, 4 brass guns were employed, very nicely bored and cast on purpose, of different lengths, but equal in all other respects, viz, in weight and bore, &c. The lengths of the bores of the guns were,

the gun	n° 1,	was 15 calibres,	length of bore	28·5 inc.
. . .	n° 2,	. 20 calibres,	. . . . .	38·4
. . .	n° 3,	. 30 calibres,	. . . .	57·7
. . .	n° 4,	. 40 calibres,	. . . .	80·2.

the calibre of each being $2\frac{1}{10}$ inches, and the medium weight of the balls 16 oz. 13 drams.

The mediums of all the experimented velocities of the balls, with which they struck the pendulous block of wood, placed at the distance of 32 feet from the muzzle of the gun, for several charges of powder, were as in the following table,

Table of Initial Velocities.

Powder.	The Guns.			
oz.	N°. 1.	N°. 2.	N°. 3.	N°. 4.
2	780	835	920	970
4	1100	1180	1300	1370
6	1340	1445	1590	1680
8	1430	1580	1790	1940
12	1436	1640	.	.
14	.	1660	.	.
16	.	.	2000	.
18	.	.	.	2200

placed in the 1st column, for all the four guns, the numbers denoting so many feet per second. Whence in general it

it appears how the velocities increase with the charges of powder, for each gun, and also how they increase as the guns are longer, with the same charge, in every instance.

By increasing the quantity of the charges continually, for each gun, it was found that the velocities continued to increase till they arrived at a certain degree, different in each gun; after which, they constantly decreased again, till the bore was quite filled with the charge. The charges of powder when the velocities arrived at their maximum or greatest state, were various, as might be expected, according to the lengths of the guns; and the weight of powder, with the length it extended in the bore, and the fractional part of the bore it occupied, are shown in the following table, of the charges for the greatest effect.

Gun, n°.	Length of the bore.	The Charge. Weight, oz.	Length. Inches.	Part of whole.
1	28·5	12	8·2	$\frac{3}{10}$
2	38·4	14	9·5	$\frac{3}{12}$
3	57·7	16	10·7	$\frac{3}{16}$
4	80·2	18	12·1	$\frac{3}{20}$

Some few experiments in this course were made to obtain the ranges and times of flight, the mediums of which are exhibited in the following table.

Guns.	Powder.	Balls. Weight.		Diam.	Eleva. gun.	Time of flight.	Range.	First veloc.
	oz.	oz.	dr.	inch.		secs.	feet.	feet.
n° 2	2	16	10	1·96	45°	21·2	5109	863
do.	2	16	5	1·96	15	9·2	4130	868
do.	4	16	8	1·96	15	9·2	4660	1234
do.	8	16	12	1·96	15	14·4	6066	1644
do.	12	16	12	1·95	15	15·5	6700	1676
n° 3	8	15	8	1·96	15	10·1	5610	1938

In this table are contained the following concomitant data, determined with a tolerable degree of precision; viz, the weight of the powder, the weight and diameter of the ball, the initial or projectile velocity, the angle of elevation of the

gun,

gun, the time in seconds of the ball's flight through the air, and its range, or the distance where it fell on the horizontal plane. From which it is hoped that some aid may be derived towards ascertaining the resistance of the medium, and its effects on other elevations, &c, and so afford some means of obtaining easy rules for the cases of practical gunnery. Though the completion of this enquiry, for want of time at present, must be referred to another work, where we may have an opportunity of describing another more extended course of experiments on this subject, which have never yet been given to the public.

Another subject of enquiry in the foregoing experiments, was, how far the balls would penetrate into solid blocks of elm wood, fired in the direction of the fibres. The annexed tablet shows the results of a few of the trials that were made with the gun n° 2, with the most frequent charges of 2, 4, and 8 ounces of powder; and the mediums of the penetrations, as placed in the last line, are found to be 7, 15, and 20 inches, with those charges. These penetrations are nearly as the numbers

Penetrations of Balls into solid Elm wood.

Powder 2	4	8 oz.
7	16·6	18·9
	13·5	21·2
		18·1
		20·8
		20·5
Means 7	15	20

2, 4, 6, or 1, 2, 3; but the charges of powder are as
2, 4, 8, or 1, 2, 4; so that the penetrations are proportional to the charges as far as to 4 ounces, but in a less ratio at 8 ounces; whereas, by the theory of penetrations, the depths ought to be proportional to the charges, or, which is the same thing, as the squares of the velocities. So that it seems the resisting force of the wood is not uniformly or constantly the same, but that it increases a little with the increased velocity of the ball. This may probably be occasioned by the greater quantity of fibres driven before the ball; which may thus increase the spring and resistance of the wood, and prevent the ball from penetrating so deep as it otherwise might do.

From a general inspection of this second course of these experiments, it appears that all the deductions and observations made on the former course, are here corroborated and strengthened, respecting the velocities and weights of the balls, and charges of powder, &c. It further appears also that the velocity of the ball increases with the increase of charge

charge only to a certain point, which is peculiar to each gun, where it is greatest; and that by further increasing the charge, the velocity gradually diminishes, till the bore is quite full of powder. That this charge for the greatest velocity is greater as the gun is longer, but yet not greater in so high a proportion as the length of the gun is; so that the part of the bore filled with powder, bears a less proportion to the whole bore in the long guns, than it does in the shorter ones; the part which is filled being indeed nearly in the inverse ratio of the square root of the empty part.

It appears that the velocity, with equal charges, always increases as the gun is longer; though the increase in velocity is but very small in comparison to the increase in length; the velocities being in a ratio somewhat less than that of the square roots of the length of the bore, but greater than that of the cube roots of the same, and is indeed nearly in the middle ratio between the two.

It appears, from the table of ranges, that the range increases in a much lower ratio than the velocity, the gun and elevation being the same. And when this is compared with the proportion of the velocity and length of gun in the last paragraph, it is evident that we gain extremely little in the range by a great increase in the length of the gun, with the same charge of powder. In fact the range is nearly as the 5th root of the length of the bore; which is so small an increase, as to amount only to about a 7th part more range for a double length of gun.—From the same table it also appears, that the time of the ball's flight is nearly as the range; the gun and elevation being the same.

It has been found, by these experiments, that no difference is caused in the velocity, or range, by varying the weight of the gun, nor by the use of wads, nor by different degrees of ramming, nor by firing the charge of powder in different parts of it. But that a very great difference in the velocity arises from a small degree in the windage: indeed with the usual established windage only, viz, about $\frac{1}{20}$ of the calibre, no less than between $\frac{1}{3}$ and $\frac{1}{4}$ of the powder escapes and is lost: and as the balls are often smaller than the regulated size, it frequently happens that half the powder is lost by unnecessary windage.

It appears too that the resisting force of wood, to balls fired into it, is not constant: and that the depths penetrated by balls, with different velocities or charges, are nearly as the logarithms of the charges, instead of being as the charges themselves, or, which is the same thing, as the square of the velocity.—Lastly, these and most other experiments, show, that

that balls are greatly deflected from the direction in which they are projected: and that as much as 300 or 400 yards in a range of a mile, or almost $\frac{1}{4}$th of the range.

We have before adverted to a 3d set of experiments, of still more importance, with respect to the resistance of the medium, than any of the former; but, till the publication of those experiments, we cannot avail ourselves of all the discoveries they contain. In the mean time however we may extract from them the three following tables of resistances, for three different sizes of balls, and for velocities between 100 feet and 2000 feet per second of time.

Table I. *Resistances to a ball of 1·965 inches diameter, and 16 oz. 13 dr. weight.*					Table II. *Resistances to a ball 2·78 in. diam. and 3lb. weight.*			Table III. *Resistances to a ball 3·55 in. diam. and 6lb. 1 oz. 8 dr. wt.*		
Vel.	Resistances.		1 Dif.	2d Dif.	Vel.	Res.	Difs.	Vel.	Res.	Difs.
feet.	lbs.	ozs.			feet.	lbs.		feet.	lbs.	
100	0·17	2¾			900	35		1200	115	
200	0·69	11	8¼	5¾	950	41	6	1250	124	9
300	1·56	25	14	6	1000	47	6	1300	133	9
400	2·81	45	20	7	1050	53	6	1350	142	9
500	4·50	72	27	8	1100	60	7	1400	152	10
600	6·69	107	35	9	1150	67	7	1450	162	10
700	9·44	151	44	10	1200	74	7	1500	172½	10½
800	12·81	205	54	12	1250	82	8	1550	184	11½
900	16·94	271	66	13	1300	91	9	1600	197	13
1000	21·88	350	79	13	1350	101	10	1650	211	14
1100	27·63	442	92	12	1400	112	11	1700	226	15
1200	34·13	546	104	11	1450	122½	10½	1750	242	16
1300	41·31	661	115	9	1500	132½	10	1800	259	17
1400	49·06	785	124	7	1550	141½	9			
1500	57·25	916	131	4	1600	150	8½			
1600	65·69	1051	135	0	1650	158	8			
1700	74·13	1186	135	−2	1700	165	7			
1800	82·44	1319	133	−5	1750	171	6			
1900	90·44	1447	128	−6	1800	176	5			
2000	98·06	1569	122							

PROBLEM I.

To determine the Resistance of the Medium against a Ball of any other size, moving with any of the Velocities given in the foregoing Tables.

The analogy among the numbers in all these tables is very remarkable and uniform, the same general laws running through them all. The same laws are also observable as in the table of resistances near the end of the 2d volume, particularly the 1st and 2d remarks immediately following that table,

table, viz, that the resistances increase in a higher proportion than the square of the velocities, with the same body; and that the resistances also increase in a rather higher ratio than the surfaces, with different bodies, but the same velocity. Yet this latter case, viz, the ratios of the resistances and of the surfaces, or of the squares of the diameters, which is the same thing, are so nearly alike, that they may be considered as equal to each other in any calculations relating to artillery practice. For example, suppose it were required to determine what would be the resistance of the air against a 24lb ball discharged with a velocity of 2000 feet per second of time. Now, by the 1st of the foregoing tables, the ball of 1·965 inches diameter, when moving with the velocity 2000, suffered a resistance of 98lb: then since the resistances, with the same velocity, are as the surfaces; and the surfaces are as the squares of the diameters; and the diameters being 1·965 and 5·6, the squares of which are 3·86 and 31·36, therefore as 3·86 : 31·36 : : 98lb : 796lb; that is, the 24lb ball would suffer the enormous resistance of 796lb in its flight, in opposition to the direction of its motion!

And, in general, if the diameter of any proposed ball be denoted by d, and r denote the resistance in the 1st table due to the proposed velocity of the 1·965 ball; then $\frac{d^2r}{3\cdot86}$ will denote the resistance with the same velocity against the ball whose diameter is d; or it is nearly $\frac{1}{4}d^2r$, which is but the 28th part greater than the former.

PROBLEM II.

To assign a Rule for determining the Resistance due to any Indeterminate Velocity of a Given Ball.

This problem is very difficult to perform near the truth, on account of the variable ratio which the resistance bears to the velocity, increasing always more and more above that of the square of the velocity, at least to a certain extent; and indeed it appears that there is no single integral power whatever of the velocity, or no expression of the velocity in one term only, that can be proportional to the resistances throughout. It is true indeed, that such an expression can be assigned by means of a fractional power of the velocity, or rather one whose index is a mixed number, viz, $2\frac{1}{10}$ or $2\cdot1$; thus $\frac{v^{2\cdot1}}{5400}$ = the resistance, is a formula in one term only, which will answer to all the numbers in the first table of resistances very nearly, and consequently, by means of the ratio of the squares

of

of the diameters of the balls, for any other balls whatever. This formula then, though serving quite well for some particular resistance, or even for constructing a complete series or table of resistances, is not proper for the use of problems in which fluxions and fluents are concerned, on account of the mixed number $2\frac{1}{10}$, in the index of the velocity v.

We must therefore have recourse to an expression in two terms, or a formula containing two integral powers of the velocity, as v^2 and v, the first and 2d powers, affected with general coefficients m and n, as $mv^2 + nv = r$ the resistance. Now, to determine the general numerical values of the coefficients m and n, we must adapt this general expression $mv^2 + nv = r$, to two particular cases of velocity, at a convenient distance from each other, in one of the foregoing tables of resistances, as the first for instance. Now, after making several trials in this way, I have found that the two velocities of 500 and 1000 answer the general purpose better than any other that has been tried. Thus then, employing these two cases, we must first make $v = 500$, and $r = 4\frac{1}{2}$ lb, its correspondent resistance; and then again $v = 1000$, and $r = 21{\cdot}88$ lb, the resistance belonging to it: this will give two equations, by which the general value of m and of n will be determined. Thus then the two equations being

$$500^2 m + 500n = 4{\cdot}5,$$

$$\text{and } 1000^2 m + 1000n = 21{\cdot}88;$$

dividing the 1st by 500, and the 2d by 1000, they are . . $\begin{cases} 500m + n = {\cdot}009, \\ 1000m + n = {\cdot}02188; \end{cases}$

the dif. of these is . . . . . $500m = {\cdot}01288$,

and therefore div. by 500, gives $m = {\cdot}00002576$;

hence $n = {\cdot}009 - 500m = {\cdot}009 - {\cdot}01288 = -{\cdot}00388 = n$.

Hence then the general formula will be ${\cdot}00002576v^2 - {\cdot}00388v = r$ the resistance nearly in avoirdupois pounds, in all cases or all velocities whatever.

Now,

Velocs. or v.	Comput. resists.	Exper. resists.
100	−·13	·17
200	+·25	·69
300	1·15	1·56
400	2·57	2·81
500	4·50	4·50
600	6·94	6·69
700	9·90	9·44
800	13·38	12·81
900	17·37	16·94
1000	21·88	21·88
1100	26·90	27·63
1200	32·44	34·18
1300	38·49	41·31
1400	45·06	49·06
1500	52·14	57·25
1600	59·74	65·69
1700	67·85	74·13
1800	76·48	82·44
1900	85·62	90·44
2000	95·28	98·06

Now, to find how near to the truth this theorem comes, in every instance in the table, by substituting for v, in this formula, all the several velocities, 100, 200, 300, &c. to 2000, these give the correspondent values of r, or the resistances, as in the 2d column of the annexed table, their velocities being in the first column; and the real experimented resistances are set opposite to them in the 3d or last column of the same. By the comparison of the numbers in these two columns together, it is seen that there are no where any great difference between them, being sometimes a little in excess, and again a little in defect, by very small differences; so that, on the whole, they will nearly balance one another, in any particular instance of the range or flight of a ball, in all degrees of its velocity, from the first or greatest, to the smallest or last. Except in the first two or three numbers at the beginning of the table, for the velocities 100, 200, 300, for which cases another theorem may be employed. Now, in these three velocities, as well as in all that are smaller, down to nothing, the theorem $\cdot00001725v^2 = r$ the resistance, will very well serve, as it brings out for the first three resistances ·17, and ·69, and 1·55¼, differing in the last only by a very small fraction.

Corol. 1. The foregoing rule $\cdot00002576v^2 - \cdot00388v = r$, denotes the resistance for the ball in the first table, whose diameter is 1·965, the square of which is 3·86, or almost 4; hence to adapt it to a ball of any other diameter d, we have only to alter the former in proportion to the squares of the diameters, by which it becomes $\frac{dd}{3\cdot86}(\cdot00002576v^2 - \cdot00388v)$ $= (\cdot00000667v^2 - \cdot001v)d^2 = (\cdot00000\frac{2}{3}v^2 - \cdot001v)d^2$, which is the resistance for the ball whose diameter is d, with the velocity v.

Corol. 2. And, in a similar manner, to adapt the theorem $\cdot00001725v^2 = r$, for the smaller velocities, to any other size

of

of ball, we must multiply it by $\frac{dd}{3 \cdot 86}$, the ratio of the surfaces, by which it becomes $\cdot 00000447 d^2 v^2 = r$.

We shall soon take occasion to make some applications in the use of the foregoing formulas, after considering the effects of such velocities in the cases of nonresistances.

PROBLEM III.

To determine the Height to which a Ball will rise, when fired from a cannon Perpendicularly Upwards with a Given Velocity, in a Nonresisting Medium, or supposing no Resistance in the Air.

By art. 73 pa. 151 vol. 2, it appears that any body projected upwards, with a given velocity, will ascend to the height due to the velocity, or the height from which it must naturally fall to acquire that velocity; and the spaces fallen being as the square of the velocities; also 16 feet being the space due to the velocity 32; therefore the space due to any proposed velocity v, will be found thus, as $32^2 : 16 :: v^2 : s$ the space, or as $64 : 1 :: v^2 : \frac{1}{64}v^2 = s$ the space, or the height to which the velocity v will cause the body to rise, independent of the air's resistance.

Exam. For example, if the first or projectile velocity, be 2000 feet per second, being nearly the greatest experimented velocity, then the rule $\frac{1}{64}v^2 = s$ becomes $\frac{1}{64} \times 2000^2 = 62500$ feet $= 11\frac{5}{6}$ miles; that is, any body, projected with the velocity 2000 feet, would ascend nearly 12 miles in height, without resistance.

Corol. Because, by art. 88 Projectiles vol 2, the greatest range is just double the height due to the projectile velocity, therefore the range, at an elevation of 45°, with the velocity in the last example, would be $23\frac{2}{3}$ miles, in a nonresisting medium. We shall now see what the effects will be with the resistance of the air.

PROBLEM IV.

To determine the Height to which a Ball projected Upwards, as in the last problem, will ascend, being Resisted by the Atmosphere.

Putting x to denote any variable and increasing height ascended by the ball; v its variable and decreasing velocity there; d the diameter of the ball, its weight being w; $m = \cdot 00000\frac{2}{3}$, and $n = \cdot 001$, the coefficients of the two terms denoting the law of the air's resistance. Then $(mv^2 - nv)d^2$, by cor. 4 to prob,

prob. 2, will be the resistance of the air against the ball in avoirdupois pounds; to which if the weight of the ball be added, then $(mv^2 - nv)d^2 + w$ will be the whole resistance to the ball's motion; this divided by w, the weight of the ball in motion, gives $\frac{(mv^2-nv)d^2+w}{w} = \frac{mv^2-nv}{w}d^2 + 1 = f$ the retarding force. Hence the general formula $v\dot{v} = 2gf\dot{x}$ (theor. 10 pa. 342 vol. 2, edit. 6) becomes $-v\dot{v} = 2g\dot{x} \times \frac{(mv^2-nv)d^2+w}{w}$, making $\dot{v}$ negative because v is decreasing, where $g = 16$ ft., and hence

$$\dot{x} = -\frac{w}{2g} \times \frac{v\dot{v}}{(mv^2-nv)d^2+w} = \frac{-w}{2gmd^2} \times \frac{v\dot{v}}{v^2 - \frac{n}{m}v + \frac{w}{md^2}}.$$

Now, for the easier finding the fluent of this, assume $v - \frac{n}{2m} = z$; then $v = z + \frac{n}{2m}$, and $v^2 = z^2 + \frac{n}{m}z + \frac{n^2}{4m^2}$, and $v\dot{v} = z\dot{z} + \frac{n}{2m}\dot{z}$, and $v^2 - \frac{n}{m}v + \frac{n^2}{4m^2} = z^2$, and $v^2 - \frac{n}{m}v = z^2 - \frac{n^2}{4m^2}$; these being substituted in the above value of $\dot{x}$, it becomes $\dot{x} =$

$$\frac{-w}{2gmd^2} \times \frac{z\dot{z} + \frac{n}{2m}\dot{z}}{z^2 - \frac{n^2}{4m^2} + \frac{w}{md^2}} = \frac{-w}{2gmd^2} \times \frac{z\dot{z} + p\dot{z}}{z^2 + \frac{w}{md^2} - p^2} = \frac{-w}{2gmd^2} \times \frac{z\dot{z} + p\dot{z}}{z^2 + q^2},$$

putting $p = \frac{n}{2m}$, and $q^2 = \frac{w}{md^2} - p^2$, or $p^2 + q^2 = \frac{w}{md^2}$.

Then the general fluents, taken by the 8th and 11th forms vol. 2 pa. 307, give $x = \frac{-w}{2gmd^2} \times [\frac{1}{2}\log.(z^2+q^2) + \frac{p}{q^2} \times$ arc to rad. q. and tan. $z] = \frac{-w}{2gmd^2} \times [\frac{1}{2}\log.(v^2 - \frac{n}{m}v + \frac{w}{md^2}) + \frac{p}{q^2} \times$ arc to rad. q and tang. $v - p]$. But, at the beginning of the motion, when the first velocity is v for instance, and the space x is $= 0$, this fluent becomes

$0 = \frac{-w}{2gmd^2} \times [\frac{1}{2}\log.(\mathrm{v}^2 - \frac{n}{m}\mathrm{v} + \frac{w}{md^2}) + \frac{p}{q^2} \times$ arc radius q tan. $\mathrm{v} - p]$. Hence by subtraction, and taking $v = 0$ for the end of the motion, the correct fluent becomes

$x = \frac{w}{2gmd^2} \times [\frac{1}{2}\log.(\mathrm{v}^2 - \frac{n}{m}\mathrm{v} + \frac{w}{md^2}) - \frac{1}{2}\log\frac{w}{md^2} + \frac{p}{q^2} \times$ (arc tan. $\mathrm{v} - p$ − arc tan. $-p$ to rad. q)].

But as part of this fluent, denoted by $\frac{p}{q^2} \times$ the dif. of the two arcs to tans. $\mathrm{v} - p$ and $-p$, is always very small in comparison

parison with the other preceding terms, they may be omitted without material error in any practical instance; and then the fluent is $x = \frac{w}{4gmd^2} \times$ hyp. log. $\frac{v^2 - \frac{n}{m}v + \frac{w}{md^2}}{\frac{w}{md^2}}$, for the utmost height to which the ball will ascend, when its motion ceases, and is stopped, partly by its own gravity, but chiefly by the resistance of the air.

But now, for the numerical value of the general coefficient $\frac{w}{4gmd^2}$, and the term $\frac{w}{md^2}$; because the mass of the ball to the diameter d, is $\cdot 5236d^3$, if its specific gravity be s, its weight will be $\cdot 5236sd^3 = w$; therefore $\frac{w}{d^2} = \cdot 5236sd$, and $\frac{w}{md^2} = 78540sd$, this divided by $4g$ or 64, it gives $\frac{w}{4gmd^2} = 1227 \cdot 2sd$ for the value of the general coefficient, to any diameter d and specific gravity s. And if we further suppose the ball to be cast iron, the specific gravity, or weight of one cubic inch of which, is $\cdot 26855$ lb, it becomes $330d$, for that coefficient; also $78540sd = 21090d = \frac{w}{md^2}$, and $\frac{n}{m} = 150$. Hence the foregoing fluent becomes $330d \times$ hyp. log.

$$\frac{v^2 - 150v + 21090d}{21090d} \text{ or } 760d \times \text{com. log.} \frac{v^2 - 150v + 21090d}{21090d},$$

changing the hyperbolic for the common logs. And this is a general expression for the altitude in feet, ascended by any iron ball, whose diameter is d inches, discharged with any velocity v feet. So that, substituting any values of d and v, the particular heights will be given, to which the balls will ascend, which it is evident will be nearly in proportion to the diameter d.

Exam. 1. Suppose the ball be that belonging to the first table of resistances, its weight being 16 oz. 13 dr. or 1·05 lb, and its diameter 1·965 inches, when discharged with the velocity 2000 feet, being nearly the greatest charge for any iron ball. The calculation being made with these values of d and v, the height ascended is found to be 2920 feet, or little more than half a mile; though found to be almost 12 miles without the air's resistance. And thus the height may be found for any other diameter and velocity.

Exam. 2. Again, for the 24 lb ball, with the same velocity 2000, its diameter being $5 \cdot 6 = d$. Here $760d = 4256$, and $\frac{v^2 - 150v + 21090d}{21090d} = \frac{38181}{1181}$, the log. of which is $1 \cdot 50958$;

theref.

theref. $1 \cdot 50958 \times 4256 = 6424 = x$ the height, being a little more than a mile.

We may now examine what will be the height ascended, considering the resistance always as the square of the velocity.

PROBLEM V.

To determine the Height ascended by a Ball projected as in the two foregoing problems; supposing the Resistance of the Air to be as the Square of the Velocity.

Here it will be proper to commence with selecting some experimented resistance corresponding to a medium kind of velocity, between the first or greatest velocity and nothing, from which to compute the other general resistances, by considering them as the squares of the velocities. It is proper to assume a near medium velocity and its resistance, because, if we assume or commence with the greatest, or the velocity of projection, and compute from it downwards, the resistances will be everywhere too great, and the altitude ascended much less than just; and, on the other hand, if we assume or commence with a small resistance, and compute from it all the others upwards, they will be much too little, and the computed altitude far too great. But, commencing with a medium degree, as for instance that which has a resistance about the half of the first or greatest resistance, or rather a little more, and computing from that, then all those computed resistances above that, will be rather too little, but all those below it too great; by which it will happen, that the defect of the one side will be compensated by the excess on the other, and the final conclusion must be near the truth.

Thus then, if we wish to determine, in this way, the altitude ascended by the ball employed in the 1st table of resistances, when projected by 2000 feet velocity; we perceive by the table, that to the velocity 2000 corresponds the resistance 98 lb; the half of this is 49, to which resistance corresponds the velocity 1400 in the table, and the next greater velocity 1500, with its resistance $57\frac{1}{4}$, which will be properest to be employed here. Hence then, for any other velocity v, in general, it will be, according to the law of the squares of the velocities, as $1500^2 : v^2 :: 57\frac{1}{4} : \frac{57\frac{1}{4}v^2}{1500^2} =$ $\cdot 000025\frac{4}{9}v^2 = av^2$, putting $a = \cdot 000025\frac{4}{9}$, which will denote the air's resistance for any velocity v, very nearly, counting from 2000.

Now let x denote the altitude ascended when the velocity is

is v, and w the weight of the ball: then, as above, av^2 is the resistance from the air, hence $av^2 + w$ is the whole resisting force, and $\frac{av^2 + w}{w} = f$ the retarding force;

$$\text{therefore } -v\dot{v} = 2gf\dot{x} = \frac{av^2 + w}{w} \times 2g\dot{x};$$

$$\text{and hence } \dot{x} = \frac{-w}{2g} \times \frac{v\dot{v}}{av^2 + w} = \frac{-w}{2ga} \times \frac{v\dot{v}}{v^2 + \frac{w}{a}};$$

the fluent of which, by form 8, is $\frac{-w}{4ga} \times$ h. log. $(v^2 + \frac{w}{a})$; which when $x = 0$, and $v = \text{v}$ the first or projectile velocity, becomes $0 = \frac{-w}{4ga} \times$ h. l. $(\text{v}^2 + \frac{w}{a})$; theref. by subtracting, the correct fluent is $x = \frac{w}{4ga} \times$ h. l. $\frac{a\text{v}^2 + w}{av^2 + w}$, the height x when the velocity is reduced to v; and when $v = 0$, or the velocity is quite exhausted, this becomes $\frac{w}{4ga} \times$ h. l. $\frac{a\text{v}^2 + w}{w}$ for the whole height to which the ball will ascend.

Ex. 1. The values of the letters being $w = 1{\cdot}05$lb, $4g = 64$, $a = {\cdot}000025\frac{4}{9}$, the last expression becomes $645 \times$ hyp. log. $\frac{\text{v}^2 + 41266}{41266}$, or $1484 \times$ com. log. $\frac{\text{v}^2 + 41266}{41266}$. And here the first vel. v being 2000, the same expression $1484 \times$ log. $\frac{\text{v}^2 + 41266}{41266}$ becomes $1484 \times$ log. of $97{\cdot}93 = 2955$ for the height ascended, on this hypothesis; which was 2920 by the former problem, being nearly the same.

Ex. 2. Supposing the same ball to be projected with the velocity of only 1500 feet. Then taking 1100 velocity, whose tabular resistance is 27·6, being next above the half of that for 1500. Hence, as $1100^2 : v^2 :: 27{\cdot}6 : {\cdot}00002375v^2 = av^2$. This value of a substituted in the theorem $\frac{w}{4ga} \times$ h. l. $\frac{a\text{v}^2 + w}{w}$, also 1500 for v, and 1·05 for w, it brings out $x = 2728$ for the height in this case.

Ex. 3. To find the height ascended by the first ball, projected with 860 feet velocity. Here taking 600, whose resistance 6·69 is a near medium; then as $600^2 : 6{\cdot}69 :: 1 : {\cdot}0000186 = a$. Hence $\frac{w}{64a} \times$ h. l. $\frac{a\text{v}^2 + w}{w} = 2334$ the height; which is less than half the range (5100) at 45° elevation, but more than half the range (4100) at 15° elevation, in pa. 161 vol. 2; being indeed nearly a medium between the two.

Ex.

Ex. 4. With the same ball, and 1640 velocity. Assume 1200, whose resistance 34·13 is nearly a medium. Then as $1200^2 : 34·13 :: 1 : ·0000237 = a$. Hence $\frac{w}{64a} \times$ h. l. $\frac{av^2 + w}{w}$ $=2854$; again less than half the range (6000) by experiment in vol. 2, even with 15° elevation.

Ex. 5. For any other ball, whose diameter is d, and its weight w, the resistance of the air being $\frac{ad^2v^2}{3·86} = \frac{d^2v^2}{150000} = bd^2v^2$ putting $b = \frac{1}{150000}$, the retarding force will be $\frac{bd^2v^2 + w}{w}$, thence $-v\dot{v} = 2g\dot{x} \times \frac{bd^2v^2 + w}{w}$, and $\dot{x} = \frac{-w}{2g} \times \frac{v\dot{v}}{bd^2v^2 + w}$, and the cor. flu. $x = \frac{w}{4gbd^2} \times$ h. l. $\frac{bd^2\mathrm{v}^2 + w}{bd^2v^2 + w} = \frac{w}{4gbd^2} \times$ h. l. $\frac{bd^2\mathrm{v}^2 + w}{w}$ for the whole height when $v = 0$. Now if the ball be a 24 pounder, whose diameter is 5·6, and its square 31·36; then $bd^2 = \frac{62·72}{300000} = ·0002091$, and $\frac{w}{4gbd^2} = \frac{24}{64bd^2} = \frac{3}{8bd^2} = 1794$; and $bd^2\mathrm{v}^2 = 836$, and $\frac{bd^2\mathrm{v}^2 + w}{w} = \frac{836 + 24}{24} = \frac{860}{24} = \frac{215}{6}$; therefore $x = 1794 \times$ h. l. $\frac{215}{6} = 1794 \times 3·57888 = 6420$, being more than double the height of that of the small ball, or a little more than a mile, and very nearly the same as in the 2d example to prob. 4.

PROBLEM VI.

To determine the Time of the Ball's ascending to the Height determined in the last prob. by the same Projectile Velocity as there given.

By that prob. $\dot{x} = \frac{-w}{2ga} \times \frac{v\dot{v}}{v^2 + \frac{w}{a}}$, ther. $\dot{t} = \frac{\dot{x}}{v} = \frac{-w}{2ga} \times \frac{\dot{v}}{v^2 + \frac{w}{a}}$; the fluent of which, by form 11, is $\frac{-w}{2ga}\sqrt{\frac{a}{w}} \times$ arc to radius 1 tang. $\frac{v}{\sqrt{\frac{w}{a}}} = \frac{-1}{2g}\sqrt{\frac{w}{a}} \times$ arc tan. $\frac{v}{\sqrt{\frac{w}{a}}}$; or by correction $t = \frac{1}{2g}\sqrt{\frac{w}{a}} \times$ (arc tang. $\frac{\mathrm{v}}{\sqrt{\frac{w}{a}}}$ − arc tang. $\frac{v}{\sqrt{\frac{w}{a}}}$), the time in general when the first velocity v is reduced to v. And when $v = 0$, or the velocity ceases, this becomes $t = \frac{1}{2g}\sqrt{\frac{w}{a}} \times$ arc to tang. $\frac{\mathrm{v}}{\sqrt{\frac{w}{a}}}$ for the time of the whole ascent.

Now,

Now, as in the last prob. $v = 2000$, $w = 1\cdot05$; $a = \cdot0000254 = \frac{229}{9000000}$. Hence $\frac{w}{a} = 41266$, and $\sqrt{\frac{w}{a}} = 203\cdot14$, and $\frac{v}{\sqrt{\frac{w}{a}}} = 9\cdot8445$ the tangent, to which corresponds the arc of $84°\ 6'$, whose length is $1\cdot4676$; then $\frac{1}{2g} \times 203\cdot14 \times 1\cdot4676 = \frac{203\cdot14 \times 1\cdot4676}{32} = 9''\cdot5$, the whole time of ascent.

Remark. The time of *freely* ascending or descending through the same height 2955 feet, that is, without the air's resistance, would be $\sqrt{\frac{2955}{16}} = \frac{1}{4}\sqrt{2955} = 13''\cdot59$; and the time of *freely* ascending, till all the velocity is lost, commencing with the same velocity 2000, would be $\frac{v}{2g} = \frac{2000}{32} = 62''\frac{1}{2} = 1'2''\frac{1}{2}$. But the time of ascending freely through the same space 2955, commencing with the same velocity 2000, would be only $1\frac{1}{2}$ seconds.

PROBLEM VII.

To determine the same as in prob. v, *taking into the account the Decrease of Density in the Air as the Ball ascends in the Atmosphere.*

In the preceding problems, relating to the height and time of balls ascending in the atmosphere, the decrease of density in the upper parts of it has been neglected, the whole height ascended by the ball being supposed in air of the same density as at the earth's surface. But it is well known that the atmosphere must and does decrease in density upwards, in a very rapid degree; so much so indeed, as to decrease in geometrical progression, at altitudes which rise only in arithmetical progression; by which it happens, that the altitudes ascended are proportional only to the logarithms of the decrease of density there. Hence it results, that the balls must be always less and less resisted in their ascent, with the same velocity, and that they must consequently rise to greater heights before they stop. It is now therefore to be considered what may be the difference resulting from this circumstance.

Now, the nature and measure of this decreasing density, of ascents in the atmosphere, has been explained and determined in prop. 76, pa. 244, &c, vol. 2. It is there shown, that if D denote the air's density at the earth's surface, and d its density at any altitude a, or x; then is $x = 63551 \times$ log. of $\frac{D}{d}$ in feet, when the temperature of the air is $55°$; and $60000 \times$ log. $\frac{D}{d}$ for the temperature of freezing cold;

we

we may therefore assume for the medium $x = 62000 \times \log. \frac{D}{d}$ for a mean degree between the two.

But to get an expression for the density d, in terms of x out of logarithms, without which it could not be introduced into the measure of the ball's resistance, in a manageable form, we find in the first place, by a neat approximate expression for the natural number to the log. of a ratio, $\frac{D}{d}$, whose terms do not greatly differ, invented by Dr. Halley, and explained in the Introduction to our Logarithms, p. 110, that $\frac{n-\frac{1}{2}l}{n+\frac{1}{2}l} \times D$ nearly, is the number answering to the log. l of the ratio $\frac{D}{d}$, where n denotes the modulus ·43429448 &c of the common logarithms. But, we before found that $x = 62000 \times \log.$ of $\frac{D}{d}$, or $\frac{x}{62000}$ is the log. of $\frac{D}{d}$, which log. was denoted by l in the expression just above, for the number whose log. is l or $\frac{x}{62000}$; substituting therefore $\frac{x}{62000}$ for l, in the expression $\frac{n-\frac{1}{2}l}{n+\frac{1}{2}l} \times D$, it gives the natural number $\frac{n - \frac{x}{124000}}{n + \frac{x}{124000}} \times D = d$, or $\frac{124000n - x}{124000n + x} = d$, the density of the air at the altitude x, putting $D = 1$ the density at the surface. Now put $124000n$ or nearly $54000 = c$; then $\frac{c-x}{c+x}$ will be the density of the air at any general height x.

But, in the 5th prob. it appears that av^2 denotes the resistance to the velocity v, or at the height x, for the density of air the same as at the surface, which is too great in the ratio of $c + x$ to $c - x$; therefore $av^2 \times \frac{c-x}{c+x}$ will be the resistance at the height x, to the velocity v, where $a =$ ·000025$\frac{4}{5}$. To this adding w, the weight of the ball, gives $av^2 \times \frac{c-x}{c+x} + w$ for the whole resistance, both from the air and the ball's mass; conseq. $\frac{av^2}{w} \times \frac{c-x}{c+x} + \frac{w}{w}$ will denote the accelerating force of the ball. Or, if we include the small part $\frac{w}{w}$ or 1, within the factor $\frac{c-x}{c+x}$, which will make no sensible difference in the result, but be a great deal simpler in the process, then is $\frac{av^2+w}{w} \times \frac{c-x}{c+x} = f$ the accelerating force.

force. Conseq. $-v\dot{v} = 2gf\dot{x} = 2g\dot{x} \times \frac{c-x}{c+x} \times \frac{av^2+w}{w}$, and hence $\frac{c-x}{c+x}\dot{x} = \frac{w}{2g} \times \frac{-v\dot{v}}{av^2+w}$, or by division, $-\dot{x} + \frac{2c}{c+x}\dot{x} = \frac{1}{32a} \times \frac{-v\dot{v}}{v^2+\frac{w}{a}}$.

Now the fluent of the first side of this equation is evidently $-x + 2c \times$ h. l. $(c+x)$; and the fluent of the latter side, the same as in prob. 5, is $\frac{-w}{64a} \times$ h. l. $(v^2 + \frac{w}{a})$; therefore the general fluential equa. is $-x + 2c \times$ h. l. $(c+x) = \frac{-w}{64a} \times$ h. l. $(v^2+\frac{w}{a})$. But, when $x = 0$, and $v = \text{v}$ the initial velocity, this becomes $0 + 2c \times$ h. l. $c = \frac{-w}{64a} \times$ h. l. $(\text{v}^2 + \frac{w}{a})$; theref. by subtraction, the correct fluents are $-x + 2c \times$ h. l. $\frac{c+x}{c} = \frac{w}{64a} \times$ h. l. $\frac{a\text{v}^2+w}{av^2+w}$, when the first velocity v is diminished to any less one v; and when it is quite extinct, the state of the fluents becomes $-x + 2c \times$ h. l. $\frac{c+x}{c} = \frac{w}{64a} \times$ h. l. $\frac{a\text{v}^2+w}{w}$, for the greatest height x ascended.

Here, in the quantity h. l. $\frac{c+x}{c}$, the term x is always small in respect of the other term c; therefore, by the nature of logarithms, the h. l. of $\frac{c+x}{c}$ is nearly $= \frac{x}{c+\frac{1}{2}x}$ or $\frac{2x}{2c+x}$; theref. the above fluents become $-x + \frac{4cx}{2c+x} = \frac{2cx-x^2}{2c+x} = \frac{2c-x}{2c+x}x = \frac{w}{64a} \times$ h. l. $\frac{a\text{v}^2+w}{w}$. Now the latter side of this equation is the same value for x as was found in the 5th problem, which therefore put $= b$; then the value of x will be easily found from the formula $\frac{2c-x}{2c+x}x = b$, by a quadratic equation. Or, still easier, and sufficiently near the truth, by substituting b for x in the numerator and the denominator of $\frac{2c-x}{2c+x}$, then $\frac{2c-b}{2c+b}x = b$, and hence $x = \frac{2c+b}{2c-b}b$, or by proportion as $2c - b : 2c + b :: b : x$; that is, only increase the value of x, found by prob. 5, in the ratio of $2c - b$ to $2c + b$.

Now, in the first example to that prob. the value of x or b was there found $= 2955$; and $2c$ being $= 108000$, theref. $2c - b = 105045$, and $2c + b = 110955$, then as $105045 : 110955 :: 2955 : 3121 =$ the value of the height x in this case, being only 166 feet, or $\frac{1}{18}$th part more than before.

Also,

Also, for the 5th example to the 5th prob. where x was = 6420; therefore as $2c-b : 2c+b$ or as 105045 : 110955 : : 6420 : 6780 the height ascended in this example, being also the 18th part more than before. And so on, for any other examples; the value of $2c$ being the constant number 108000.

PROBLEM VIII.

To determine the Time of a Ball's Ascending, considering the Decreasing Density of the Air as in the last prob.

The fluxion of the time is $\dot{t} = \frac{\dot{x}}{v}$. But the general equation of the fluxions of the space x and velocity v, in the last prob. was $\frac{c-x}{c+x}\dot{x} = \frac{w}{32} \times \frac{-v\dot{v}}{av^2+w}$; ther. $\dot{x} = \frac{w}{32} \times \frac{c+x}{c-x} \times \frac{-v\dot{v}}{av^2+w}$; hence $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{w}{32} \times \frac{c+x}{c-x} \times \frac{-\dot{v}}{av^2+w}$. But x, which is always small in respect of c, is nearly $= b$ as determined in the last problem; theref. $\frac{c+b}{c-b}$ may be substituted for $\frac{c+x}{c-x}$ without sensible error; and then $\dot{t}$ becomes $= \frac{w}{32} \times \frac{c+b}{c-b} \times \frac{-\dot{v}}{av^2+w}$. Now, this fluxion being to that in prob. 6, in the constant ratio of $c-b$ to $c+b$, their fluents will be also in the same constant ratio. But, by the last prob. $c = 54000$, and $b = 2955$ for the first example in prob. 5; therefore $c - b = 51045$, and $c + b = 56955$, also, the time in problem 6 was 9″·91; therefore as 51045 : 56955 : : 9″·91 : 11″·04 for the time in this case, being 1″·13 more than the former, or nearly the 9th part more; which is nearly the double, or as the square of the difference, in the last prob. in the height ascended.

PROBLEM IX.

To determine the circumstances of Space, Time, and Velocity, of a Ball Descending through the Atmosphere by its own Weight.

It is here meant that the balls are at least as heavy as cast iron, and therefore their loss of weight in the air insensible; and that their motion commences by their own gravity from a state of rest. The first object of enquiry may be, the utmost degree of velocity any such ball acquires by thus descending. Now it is manifest that the ball's motion is commenced, and uniformly increased, by its own weight, which is its constant urging force, being always the same, and producing an equal increase of velocity in equal times, excepting for the diminution of motion by the air's resistance. It is also evident that

 this

this resistance, beginning from nothing, continually increases, in some ratio, with the increasing velocity of the ball. Now, as the urging force is constantly the same, and the resisting force always increasing, it must happen that the latter will at length become equal to the former: when this happens, there can afterwards be no further acceleration of the motion, the impelling force and the resistance being equal, and the ball must ever after descend with a uniform motion. It follows therefore that, to answer the first enquiry, we have only to determine when or what velocity of the ball will cause a resistance just equal to its own weight.

Now, by inspecting the tables of resistances preceding prob. 1, particularly the 1st of the three tables, the weight of the ball being 1·05lb, we perceive that the resistance increases in the 2d column, till 0·69 opposite to 200 velocity, and 1·56 answering to 300 velocity, between which two the proposed resistance 1·05, and the correspondent velocity, fall. But, in two velocities not greatly different, the resistances are very nearly proportional to the squares of the velocities. Therefore, having given the velocity 200 answering to the resistance 0·69, to find the velocity answering to the resistance 1·05, we must say, as $0·69 : 1·05 :: 200^2 : v^2 = 60870$, theref. $v = \sqrt{60870} = 246$, is the greatest velocity this ball can acquire; after which it will descend with that velocity uniformly, or at least with a velocity nearly approaching to 246.

The same greatest or uniform velocity will also be directly found from the rule $·00001725v^2 = r$, near the end of problem 2, where r is the resistance to the velocity v, by making $1·05 = r$; for then $v^2 = \frac{1·05}{·00001725} = 60870$, the same value for v^2 as before.

But now, for any other weight of ball; as the weights of the balls increase as the cubes of their diameters, and their resistances, being as the surfaces, increase only as the squares of the same, which is one power less; and the resistances being also in this case, as the squares of the velocities, we must therefore increase the squares of the velocity in the ratio of the diameters of the balls; that is, as $1·965 : d :: 246^2 : \frac{246^2}{1·965}d = v^2$, and hence $v = 246\sqrt{\frac{d}{1·965}} = 175\frac{1}{2}\sqrt{d}$.

If we take here the 3lb ball, belonging to the 2d table of resistances, whose diameter d is $= 2·80$; then $\sqrt{2·80} = 1·673$, and $175\frac{1}{2} \times 1·67 = 294$, is the greatest or uniform velocity, with which the 3lb ball will descend. And if we take the 6lb ball, whose diameter is 3·53 inches, as in the 3d table of resistances:

resistances: then $\sqrt{3\cdot53} = 1\cdot88$, and $175\frac{1}{4} \times 1\cdot88 = 330$, being the greatest velocity that can be acquired by the 6 lb ball, and with which it will afterwards uniformly descend. For a 9lb ball, whose diameter is 4·04, the velocity will be $175\frac{1}{4} \times 2\cdot01 = 353$. And so on for any other size of iron ball, as in the following table. Where the first column contains the weight of the balls in lbs; the 2d their diameters in inches; the 3d their velocities to which they nearly approach, as a limit, and therefore called their terminal or last velocities, with which they afterward descend uniformly; and the 4th or last column the heights due to those velocities, or the heights from which the balls must descend in vacuo to acquire them.

Wt. lbs.	Diam. inch.	Term. Veloc. feet.	Height due to v, feet.
1	1·94	244	930
2	2·45	275	1182
3	2·80	294	1360
4	3·08	308	1482
6	3·53	330	1701
9	4·04	353	1958
12	4·45	370	2139
18	5·09	396	2450
24	5·60	415	2691
32	6·17	436	2970
36	6·41	444	3080
42	6·75	456	3249

But it is manifest that the balls can never attain exactly to these velocities in any finite time or descent, being only the limits to which they continually approach, without ever really reaching, though they arrive very nearly at them in a short space of time; as will appear by the following calculation.

To obtain general expressions for the space descended, and the time of the descent, in terms of the velocity v: put $x =$ any space descended, $t =$ its time, and v the velocity acquired, the weight of the ball $w = 1\cdot05$ lb. Now, by the theorem near the end of prob. 2, which is the proper rule for this case, the velocity being small, $\cdot000017\,25v^2 = cv^2$ is the resistance due to the velocity v; theref. $w - cv^2$ is the impelling force, and $\frac{w - cv^2}{w} = f$ the accelerating force; conseq. $v\dot{v}$ or $2gf\dot{x} = 2g\dot{x} \times \frac{w - cv^2}{w}$, and $\dot{x} = \frac{w}{2g} \times \frac{v\dot{v}}{w - cv^2}$, the correct fluent of which, by the 8th form, is $x = \frac{w}{4gc} \times$ h. l. $\frac{w}{w - cv^2}$, the general value of the space x descended.

Here it appears that the denominator $w - cv^2$ decreases as v increases; conseq. the whole value of x, the descent, increases with v, till it becomes infinite, when the resistance cv^2 is $= w$ the weight of the ball, when the motion becomes

 uniform,

uniform, as before remarked. We may however easily assign the value of x a little before the velocity becomes uniform, or before cv^2 becomes $= w$. Thus, when $cv^2 = w$, then $v = 246$, as found in the beginning of this problem. Assume therefore v a little less than that greatest velocity, as for instance 240: then this value of v substituted in the general formula for x above deduced, gives $x = 2781$ feet, a little before the motion becomes uniform, or when the velocity has arrived at 240, its maximum being 246.

In like manner is the space to be computed that will be due to any other velocity less than the greatest or terminal velocity. On the contrary, to find the velocity due to any proposed space x, from the formula $x = \frac{w}{4gc} \times \text{h. l.} \frac{w}{w - cv^2}$. Here x is given, to find v. First then $\frac{4gcx}{w} = \text{h. l.} \frac{w}{w - cv^2}$; take therefore the number to the hyp. log. of $\frac{4gcx}{w}$, which number call N; then $N = \frac{w}{w - cv^2}$; conseq. $Nw - Ncv^2 = w$, and $Nw - w = Ncv^2$, and $v = \sqrt{\frac{N-1}{Nc}}w$, a general theorem for the value of v due to any distance x. Suppose, for instance, x is 1000. Now $4g$ being $= 64$, $w = 1{\cdot}05$, and $c = {\cdot}00001725$; theref. $\frac{4gcx}{w} = 1{\cdot}0514$, and the natural number belonging to this, considered as an hyp. log. is $2{\cdot}8617 = N$; hence then $v = \sqrt{\frac{N-1}{Nc}}w = 199$, is the velocity due to the space 1000, or when the ball has descended 1000 feet.

Again, for the time t of descent: here $\dot{t} = \frac{\dot{x}}{v}$, but $\dot{x} = \frac{w}{2g} \times \frac{v\dot{v}}{w - cv^2}$, as found above, theref. $\dot{t} = \frac{w}{2g} \times \frac{\dot{v}}{w - cv^2}$, the fluent of which is $\frac{1}{4g}\sqrt{\frac{w}{c}} \times \text{h. l.} \frac{\sqrt{\frac{w}{c}} + v}{\sqrt{\frac{w}{c}} - v}$, the general value of the time t for any value of the velocity v; which value of t evidently increases as the denominator $\sqrt{\frac{w}{c}} - v$ decreases, or as the velocity v increases; and consequently the time is infinite when that denominator vanishes, which is when $v = \sqrt{\frac{w}{c}}$, or $cv^2 = w$, the resistance equal to the ball's weight, being the same case as when the space x becomes infinite, as above remarked. But, like as was done for

for the distance x as above, we may here also find the value of t corresponding to any value of v, less than its maximum 246, and consequently to any value of x, as when v is 240 for instance, or $x = 2781$, as determined above. Now, by substituting 240 for v, in the general formula

$$t = \frac{1}{4g}\sqrt{\frac{w}{c}} \times \text{h. l.} \frac{\sqrt{\frac{w}{c}} + v}{\sqrt{\frac{w}{c}} - v}, \text{ it brings out } t = 16''\cdot575;$$ so

that it would be nearly $16\frac{1}{2}$ seconds when the velocity arrives at 240, or a little less than the maximum or uniform degree, viz, 246, or when the space descended is 2781 feet.

Also, to determine the time corresponding to the same, when the descent is 1000 feet, or the velocity 199: find the value of $\frac{1}{4g}\sqrt{\frac{w}{c}} = \frac{1}{64}\sqrt{\frac{1\cdot05}{\cdot00001725}} = \frac{246}{64} = \frac{123}{32}$. Then

$$\frac{\sqrt{\frac{w}{c}} + v}{\sqrt{\frac{w}{c}} - v} = \frac{246 + 199}{246 - 199} = \frac{445}{47};$$ the hyp. log. of which is 2·2479.

Hence $2\cdot2479 \times \frac{123}{32} = 8''\cdot64$, the time of descending 1000 feet, or when the velocity is 199.

See other speculations on this problem, in the 2d volume, prob. 22, as determined from theory, viz, without using the experimented resistance of the air.

PROBLEM X.

To determine the Circumstances of the Motion of a Ball projected Horizontally in the Air; abstracted from its Vertical Descent by its Gravitation.

Putting d for the diameter, and w the weight of the ball, v the velocity of projection, and v the velocity of the ball after having moved through the space x. Then, by corol. 1 to prob. 2, if the velocity is considerable, such as usual in practice, the resistance of the ball, moving with the velocity v, is $(mv^2 - nv)d^2$, and therefore $\frac{mv^2 - nv}{w}d^2$ is the retardive force f; hence the common formula $v\dot{v} = 2gf\dot{x}$, is $-v\dot{v} = 32\dot{x} \times \frac{mv^2 - nv}{w}d^2$, and theref. $\dot{x} = \frac{w}{32d^2} \times \frac{-v\dot{v}}{mv^2 - nv} = \frac{w}{32d^2} \times \frac{-\dot{v}}{mv - n} = \frac{w}{32d^2m} \times \frac{-\dot{v}}{v - \frac{n}{m}}$, the fluent of which is obviously $\frac{w}{32md^2} \times -$ hyp. log. of $v - \frac{n}{m}$, and by the correction by the first

first velocity V, it becomes $x = \frac{w}{32md^2} \times$ h. log. $\frac{V - \frac{n}{m}}{v - \frac{n}{m}}$, the general formula for the distance passed over in terms of the velocity.

Now, for an application, let it be required first, to determine in what space a 24 lb ball will have its velocity reduced from 1780 feet to 1500, that is, losing 280 feet of its first velocity. Here, $d = 5{\cdot}6$, $w = 24$, $V = 1780$, and $v = 1500$; also $\frac{n}{m} = 150$. Hence $\frac{w}{32md^2} = 3587{\cdot}4$, then $x = 3587{\cdot}4 \times$ h. l. $\frac{V - 150}{v - 150} = 3587{\cdot}4 \times$ h. l. $\frac{1630}{1350} = 3587{\cdot}4 \times$ h. l. $\frac{163}{135} =$ 676 feet, the space passed over when the ball has lost 280 feet of its motion.

Again, to find with what velocity the same ball will move, after having described 1000 feet in its flight. The above theorem is x or $1000 = 3587{\cdot}4 \times$ h. l. $\frac{V - 150}{v - 150} = 3587{\cdot}4 \times$ h. l. $\frac{1630}{v - 150}$, or $\frac{10000}{35874} =$ h. l. $\frac{1630}{v - 150}$; but the number to the hyp. log. $\frac{10000}{35874}$ is $1{\cdot}7416 = N$ suppose; then $N = \frac{1630}{v - 150}$, and $Nv - 150N = 1630$, or $Nv = 1630 + 150N$, and $v = \frac{1630}{N} + 150 = 936 + 150 = 1086$, the velocity when the ball has moved 1000 feet.

Next, to find a theor. for the time of describing any space, or destroying any velocity: Here $\dot t = \frac{\dot x}{v} = \frac{w}{32md^2} \times \frac{-\dot v \div v}{v - \frac{n}{m}}$, the fluent of which, by the 9th form, is $t = \frac{w}{32md^2} \times \frac{m}{n} \times$ h. l. $\frac{v - \frac{n}{m}}{v - \frac{n}{m}} = \frac{w}{32nd^2} \times$ h. l. $\frac{v}{v - \frac{n}{m}}$, and by correction $t = \frac{w}{32nd^2} \times$ (h. l. $\frac{V}{V - \frac{n}{m}}$ − h. l. $\frac{v}{v - \frac{n}{m}}$) $= \frac{w}{32nd^2} \times$ hyp. log. $\frac{V - 150}{v - 150} \cdot \frac{v}{V}$, putting V for the first velocity, and 150 for $\frac{n}{m}$ its value, as before.

Now, to take for an example the same 24lb ball, and its projected velocity 1780, as before; let it be required to find in what time this velocity will be reduced to 786. Here then $V = 1780$, $v = 786$, $w = 24$, $d = 5{\cdot}6$, $d^2 = 31{\cdot}36$, $n = {\cdot}001$;

hence

hence $\frac{w}{32nd^2} = \frac{750}{31\cdot36} = 23\cdot916$; and $\frac{V-150}{v-150} \cdot \frac{v}{V} = \frac{1630}{636} \times \frac{786}{1780} = \frac{21142}{18868}$, the hyp. log. of which is $\cdot1099$; then $23\cdot916 \times \cdot1099 = 2''\cdot628$, the time required.

For another example, let it be required to find when the velocity will be reduced to 1000, or 780 destroyed. Here $v = 1000$, and all the other quantities as before. Then $\frac{V-150}{v-150} \times \frac{v}{V} = \frac{1630}{850} \times \frac{1000}{1780} = \frac{1630}{1513}$, the hyp. log. of which is $\cdot07449$; theref. $23\cdot916 \times \cdot07449 = 1''\cdot78$, is the time sought.

On the other hand, if it be required to find what will be the velocity after the ball has been in motion during any given time, as suppose 2 seconds, we must reverse the calculation thus: $t = 2''$ being $= \frac{w}{32nd^2} \times$ h. l. $\frac{V-150}{v-150} \cdot \frac{v}{V} = 23\cdot916 \times$ h. l. $\frac{V-150}{v-150} \cdot \frac{v}{V}$; theref. $\frac{2}{23\cdot916} = \cdot083626$ is the hyp. log. of $\frac{V-150}{v-150} \cdot \frac{v}{V}$, the number answering to which is $1\cdot08725 = N$ suppose, that is, $N = \frac{V-150}{v-150} \cdot \frac{v}{V}$. Hence $NVv - 150NV = Vv - 150v$, and $v = \frac{150NV}{150 + NV - V} = \frac{290290}{305\cdot305} = 951$, the velocity at the end of 2 seconds.

The foregoing calculations serve only for the higher velocities, such as exceed 200 or 300 feet per second of time. But, for those that are below 300, the rule is simpler, as the resistance is then, by cor. 2 prob. 2, $\cdot0000447d^2v^2 = cd^2v^2$, where d denotes the diameter of any ball. Hence then, employing the same notation as before, $\frac{cd^2v^2}{w} = f$, and $-\dot{v}v = 32f\dot{x} = 32\dot{x} \times \frac{cd^2v^2}{w}$; theref. $\dot{x} = \frac{w}{32cd^2} \times \frac{-\dot{v}}{v}$, the correct fluent of which is $x = \frac{w}{32cd^2} \times$ h. l. $\frac{V}{v}$.

Now, for an example, suppose the first velocity to be $300 = V$, and the last $v = 100$, for a 24 lb ball. Then $w = 24$, $d = 5\cdot6$, $d^2 = 31\cdot36$, $c = \cdot0000447$; therefore $\frac{w}{32cd^2} = \frac{3}{125\cdot44c} = 5350$; and $\frac{V}{v} = \frac{300}{100} = 3$, the hyp. log. of which is $1\cdot0986$; theref. $1\cdot0986 \times 5350 = 5878 = x$, is the distance.—If the first velocity be only $200 = V$; then $\frac{V}{v} = 2$, the hyp. log. of which is $\cdot69315$, therefore $\cdot69315 \times 5350 = 3708 = x$, the distance.

And

And conversely, to find what velocity will remain after passing over any space, as 4000 feet, the first velocity being $\mathrm{v} = 200$. Here the hyp. log. of $\frac{\mathrm{v}}{v}$ is $\frac{x}{5350} = \frac{4000}{5350} = \frac{400}{535}$ $= \frac{80}{107} = \cdot 74766$, the natural number of which is $2\cdot1120$, that is, $2\cdot112 = \frac{\mathrm{v}}{v}$; therefore $v = \frac{\mathrm{v}}{2\cdot112} = \frac{200}{2\cdot112} = 94\cdot7$, the velocity.

Again, for the time t: since $\dot{x} = \frac{w}{32cd^2} \times \frac{-\dot{v}}{v}$, therefore $\dot{t} = \frac{\dot{x}}{v} = \frac{w}{32cd^2} \times \frac{-\dot{v}}{v^2}$, the correct fluent of which is $t = \frac{w}{32cd^2} \times (\frac{1}{v} - \frac{1}{\mathrm{v}}) = \frac{w}{32cd^2} \times \frac{\mathrm{v} - v}{\mathrm{v}v}$.—So, for example, if $\mathrm{v} = 300$, and $v = 100$; then $\frac{\mathrm{v} - v}{\mathrm{v}v} = \frac{200}{30000} = \frac{2}{300}$; then $\frac{w}{32cd^2}$ or $5350 \times \frac{2}{300} = 35''\frac{2}{3} = t$, the time of reducing the 300 velocity to 100, or of passing over the space 5878 feet.

And, reversing, to find the velocity v, answering to any given time t: Since $t = \frac{w}{32cd^2} \times (\frac{1}{v} - \frac{1}{\mathrm{v}}) = 5350 \times (\frac{1}{v} - \frac{1}{\mathrm{v}})$, theref. $v = \frac{5350\mathrm{v}}{5350 + t\mathrm{v}}$. Here, if t be given $= 30''$, and $\mathrm{v} = 300$; then $v = \frac{5350\mathrm{v}}{5350 + 9000} = \frac{535}{1435} \times 300 = \frac{32100}{287} =$ 112, the velocity sought.

Corol. The same form of theorem, $x = \frac{w}{32cd^2} \times$ h. l. $\frac{\mathrm{v}}{v}$, as above, is brought out for small velocities, will also serve for the higher ones, if we employ the medium resistance between the two proposed velocities, as was done in prob. 5. Thus, as in the first example of this problem, where the two velocities are 1780 and 1500, the resistance due to the velocity 1700, in the first table of resistances, being 74·13, say as $1700^2 : 1780^2 :: 74\cdot13 : 81\cdot27$, the resistance due to the velocity 1780; then the mean between 81·27 and 57·25, due to 1500 velocity, is 69 26, or rather take $69\frac{1}{4}$. Again, as $\sqrt{65\cdot7} : \sqrt{69\frac{1}{2}} :: 1600 : 1646$, the velocity due to the medium resistance $69\frac{1}{4}$. Hence, as in prob. 5, as $1646^2 : v^2 :: 69\frac{1}{2} : 0000256\bar{5}v^2 =$ suppose av^2, the resistance due to any velocity v, between 1780 and 1500, for the 1·05 lb ball. And, as $1\cdot965^2 : 5\cdot6^2 :: av^2 : 8\cdot124av^2 = \cdot00020838v^2 = bv^2$ suppose, the resistance due to the same velocity with the 24 lb ball. Therefore $\frac{bv^2}{24} = f$, and $-v\dot{v} = 32f\dot{x} = \frac{4}{3}bv^2\dot{x}$, and

$\dot{x} =$

$\dot{x} = \frac{-3\dot{v}}{4bv}$, the correct fluent of which is $\frac{3}{4b} \times \text{h. l.} \frac{V}{v} = \frac{3}{4b} \times \text{h. l.} \frac{178}{150} = \frac{3}{4b} \times \text{h. l.} \frac{89}{75} = 3600 \times \cdot171148 = 616$ the velocity sought.

PROBLEM XI.

To determine the Ranges of Projectiles in the Air.

To determine, by theory, the trajectory a projectile describes in the air, is one of the most difficult problems in the whole course of dynamics, even when assisted by all the experiments that have hitherto been made on this branch of physics; and is indeed much too difficult for this place, in the full extent of the problem: the consideration of it must therefore be reserved for another occasion, when the nature of the air's resistance can be more amply discussed. Even the solutions of Newton, of Bernoulli, of Euler, of Borda, &c, &c, after the most elaborate investigations, assisted by all the resources of the modern analysis, amount to no more than distant approximations, that are rendered nearly useless, even to the speculative philosopher, from the assumption of a very erroneous law of resistance in the air, and much more so to the practical artillerist, both on that account, and from the very intricate process of calculation, which is quite inapplicable to actual service. The solution of this problem requires, as an indispensable datum, the perfect determination by experiment of the nature and laws of the air's resistance at different altitudes, to balls of different sizes and densities, moving with all the usual degrees of celerity. Unfortunately however, hardly any experiments of this kind have been made, excepting those which on some occasions have been published by myself, as in my Tracts of 1786, as well as in my Dictionary, some few of which are also given in the 2d vol. of this course, art. 105, with some practical inferences. And though I have many more yet to publish, of the same kind, much more extensive and varied, I cannot yet undertake to pronounce that they are fully adequate to the purpose in hand.

All that can be here done then, in the solution of the present problem, besides what is delivered in the 2d volume, is to collect together some of the best practical rules, founded partly on theory, and partly on practice. 1. In the first place then, it may be remarked, that the initial or first velocity of a ball may be directly computed by prob. 17, near the end of our 2d volume; having given the dimensions of the piece,

the

the weight of the ball, and the charge of powder. Or otherwise, the same may be made out from the table of experimented ranges and velocities in pa. 161 of that volume, by this rule, that the velocities to different balls, and different charges of powder, are as the square roots of the weights of the powder directly, and as the square roots of the weights of the balls inversely. Thus, if it be enquired, with what velocity a 24 lb ball will be discharged by 8 lb of powder. Now it appears in the table, that 8 ounces of powder discharge the 1 lb ball with 1640 feet velocity; and because 8 lb are = 128 ounces; therefore by the rule, as $\sqrt{\frac{8}{1}} : \sqrt{\frac{128}{24}} :: 1640 : 1640\sqrt{\frac{16}{24}} = 1640\sqrt{\frac{2}{3}} = 1339$, the velocity sought. Or otherwise, by rule 1 p. 162 of the 2d vol. as $\sqrt{24} : \sqrt{16} :: 1600 : 1306$, the same velocity nearly. But when the charges bear the same ratio to one another as the weight of the balls, that is when the pieces are said to be alike charged, then the velocities will be equal. Thus, the 1 lb ball by the 2 oz charge, being the 8th part of the weight, and the 24 lb ball, with 3 lb of powder, its 8th part also, will have the same velocity, viz, 860 feet. In like manner, the 1230 tabular velocity, answering to 4 oz of powder, the 4th part of the ball, will equally belong to the 24 lb ball with 6 lb of powder, being its 4th part, and the tabular velocity 1640, answering to the 8 oz charge, which is $\frac{1}{2}$ the weight of ball, will equally belong to the 24 lb ball with 12 lb of powder, being also the $\frac{1}{2}$ of its weight.

2. By prob. 9 will be found what is called the *terminal velocity*, that is, the greatest velocity a ball can acquire by descending in the air; indeed a table is there given of the several terminal velocities belonging to the different balls, with the heights, in an annexed column, due to those velocities in vacuo, that is the heights from which a body must fall in vacuo, to acquire those velocities.

3. Given the initial velocity, to find the elevation of the piece to have the greatest range, and the extent of that range. These will be found by means of the annexed table, altered

from

from Professor Robison's, in the Encyclopædia Britannica, and founded on an approximation of Sir I. Newton's. The numbers in the first column, multiplied by the terminal velocity of the ball, give the initial velocity; and the numbers in the last column, being multiplied by the height, give the greatest ranges; the middle column showing the elevations to produce those ranges.

Table of Elevations giving the Greatest Range.

Initial vel. div. by v.	Elevation.	Range div. by a.
0·6910	44° 0′	0·3914
0·9445	43 15	0·5850
1·1980	42 30	0·7787
1·4515	41 45	0·9724
1·7050	41 0	1·1661
1·9585	40 15	1·3598
2·2120	39 30	1·5535
2·4655	38 45	1·7472
2·7190	38 0	1·9409
2·9725	37 15	2·1346
3·2260	36 30	2·3283
3·4795	35 45	2·5220
3·7330	35 0	2·7157
3·9865	34 15	2·9094
4·2400	33 30	3·1031
4·4935	32 45	3·2968
4·7470	32 0	3·4905
5·0000	31 15	3·6842

To use this table then, divide the given initial velocity by the terminal velocity peculiar to the ball, found in the table in prob. 9, and look for the quotient in the first column here annexed. Against this, in the 2d column will be found the elevation to give the greatest range; and the number in the 3d column multiplied by a, the altitude due to the terminal velocity, also found in the table in problem 9, will give the range, nearly.

Ex. 1. Let it be required to find the greatest range of a 24 lb ball, when discharged with 1640 feet velocity, and the corresponding angle to produce that range. By the table in prob. 9, the terminal velocity of the 24 lb ball is 415, and its producing altitude 2691: hence $\frac{1640}{415} = 3{\cdot}95$, nearly equal to 3·9865 in the 1st column of our table, to which corresponds the angle 34° 15′, being the elevation to produce the greatest range; and the corresponding number 2·9094, in the 3d column, multiplied by 2691, gives 7829 feet, for the greatest range, being nearly a mile and a half.

Exam. 2. In like manner, the same ball discharged with the velocity 860 feet, will have for its greatest range 3891 feet, or nearly $\frac{3}{4}$ of a mile, and the elevation producing it 39° 55′.

These examples, and indeed the whole table in the 9th problem

problem, are only adapted to the use of cannon balls. But it is not usual, and indeed not easily practicable, to discharge cannon shot at such elevations, in the British service, that practice being the peculiar office of mortar shells. On this account then it will be necessary to make out a table of terminal velocities, and altitudes due to them, for the different sizes of such shells. The several kinds of these in present use, are denominated from the diameters of their mortar bores in inches, being the five following, viz, the 4·6, the 5·8, the 8, the 10, and the 13 inch mortars, as in the first column of the following table. But the outer diameters of the shells are somewhat smaller, to leave a little room or space as windage, as contained in the 2d column.

Table of dimensions, &c, of Mortar Shells.

Diam. of mortar	Diam. of shells	Wt. of shells filled	Weight of equal solid	Ratio of shell to solid	Terminal velocity	Alt. a due to veloc.
inches	inches	lbs	lbs		feet	feet
4·6	4·53	9	$12\frac{3}{4}$	1·42	314	1541
5·8	5·72	18	$25\frac{1}{2}$	1·42	352	1936
8	7·90	47	67	1·42	414	2678
10	9·84	$91\frac{1}{2}$	130	1·42	462	3335
13	12·80	201	286	1·42	527	4340

The 3d column contains the weight of each shell when the hollow part is filled with powder: the diameter of the hollow is usually $\frac{7}{10}$ of that of the mortar: the weight of the shells empty and when filled, with other circumstances, may be seen at Quest. 53, pa. 265, vol. 2. On account of the vacuity of the shell being filled only with gunpowder, the weight of the whole so filled, and contained in column 3, is much less than the weight of the same size of solid iron, and the corresponding weights of such equal solid balls are contained in col. 4. The ratio of these weights, or the latter divided by the former, occupies the 5th column.

Now because the loaded or filled shells are of less specific gravity, or less heavy, than the equal solid iron balls, in the ratio of 1 to 1·42, as in column 5, the former will have less power or force to oppose the resistance of the air, in that same proportion, and the terminal or greatest velocity, as determined in the 9th prob. will be correspondently less. Therefore, instead of the rule there given, viz, $175·5\sqrt{d}$, for that velocity, the rule must now be $175·5\sqrt{\frac{d}{1·42}} = 147·3\sqrt{d} = v$, the

the diameter of the shell being d; that is, the terminal velocities will be all less in the ratio of 147·3 to 175·5. Now, computing these several velocities by this rule, to all the different diameters, they are found as placed in the 6th col.; and in the 7th or last column are set the altitudes which would produce these velocities in vacuo, as computed from this theorem $\frac{vv}{64}$.

Having now obtained these terminal velocities, and their producing altitudes, for the shells, we can, from them and the former table of ranges and elevations, easily compute the greatest range, and the corresponding angle of elevation, for any mortar and shell, in the same way as was done for the balls in this problem. Thus, for example, to find the greatest range and elevation, for the 13 inch shell, when projected with the velocity of 2000 feet per second, being nearly the greatest velocity that balls can be discharged with. Now, by the method before used, $\frac{2000}{527} = 3{\cdot}796$; opposite to this, found in the first column of the table of ranges, corresponds 34° 49′ for the elevation in the 2d column, and the number 2·764 in the 3d column; this multiplied by the altitude 4340, gives 11995 feet, or more than $2\frac{1}{4}$ miles, for the greatest range.

This however is much short of the distance which it is said the French have lately thrown some shells at the siege of Cadiz, viz, 3 miles, which it seems has been effected by means of a peculiar piece of ordnance, and by loading or filling the cavity of the shell with lead, to render it heavier, and thus make it fitter to overcome the resistance of the air. Let us then examine what will be the greatest range of our 13 inch shell, if its usual cavity be quite filled with lead when discharged, with the projectile velocity of 2000 feet.

Now the diameter of the cavity, being about $\frac{7}{10}$ of that of the mortar 13, will be nearly 9 inches. And the weight of a globe of lead of this diameter is 139·3lb; which added to 187·8, the weight of the shell empty, gives 327lb, the whole weight of the shell when the cavity is filled with lead, which was found 286 when supposed all of solid iron, their ratio or quotient is ·8783. Then, as before, the theorem will be $175{\cdot}5\sqrt{\frac{d}{{\cdot}8783}} = 187{\cdot}3\sqrt{d}$ for the terminal velocity; which, when $d = 12{\cdot}8$, becomes 670 for the terminal velocity; therefore its producing altitude is $\frac{670^2}{64} = 7014$. Then, by the same method as before, $\frac{2000}{670} = 2{\cdot}985$; which number found

found in the first column of the table of ranges, the opposite number in the 2d col. is 37° 15′ for the elevation of the piece, and in the 3d column 2·14, multiplied by 7014, gives 15010 feet, or nearly 3 miles. So that our 13 inch shells, discharged at an elevation of about 37¼ degrees, would range nearly the distance mentioned by the French, when filled with lead, if they can be projected with so much as 2000 feet velocity, or upwards. This however it is thought cannot possibly be effected by our mortars; and that it is therefore probable the French, to give such a velocity to those shells, must have contrived some new kind of large cannon on the occasion.

4. Having shown in the preceding articles and problems, how, from our theory of the air's resistance, can be found, first the initial or projectile velocity of shot and shells; 2dly, the terminal velocity, or the greatest velocity a ball can acquire by descending by its own weight in the air; 3dly, the height a ball will ascend to in the air, being projected vertically with a given velocity, also the time of that ascent; 4thly, the *greatest* horizontal ranges of given shot, projected with a given velocity; as also the particular angle of elevation of the piece, to produce that greatest range. It remains then now to enquire, what laws and regulations can be given respecting the ranges, and times of flight, of projects made at other angles of elevation.

Relating to this enquiry, the Encyclopædia Britannica mentions the two following rules: 1st. "Balls of equal density, projected with the same elevation, and with velocities which are as the square roots of their diameters, will describe similar curves. This is evident, because, in this case, the resistance will be in the ratio of their quantities of motion; therefore all the homologous lines of the motion will be in the proportion of the diameters." But though this may be nearly correct, yet it can hardly ever be of any use in practice, since it is usual and proper to project small balls, not with a less, but with a greater velocity, than the larger ones. 2dly, the other rule is, "If the initial velocities of balls, projected with the same elevation, be in the *inverse* subduplicate ratio of the whole resistances, the ranges, and all the homologous lines in their track, will be inversely as those resistances." This rule will come to the same thing, as having the initial velocities in the inverse ratio of the diameters, as distant perhaps from fitness as the former. Two tables are next given in the same place, for the comparison of ranges and projectile velocities, the numbers in which appear to be much wide of the truth, as depending on very erroneous effects of the resistance. Most of the accompanying remarks, however,

however, are very ingenious, judicious, and philosophical, and very justly recommending the making and recording of good experiments on the ranges and times of flight of projects, of various sizes, made with different velocities, and at various angles of elevation.

Besides the above, we find rules laid down by Mr. Robins and Mr. Simpson, for computing the circumstances relating to projectiles as affected by the resistance of the air. Those of the former respectable author, in his ingenious Tracts on Gunnery, being founded on a quantity which he calls *F*, (answering to our letter *a* in the foregoing pages), I find to be almost uniformly double of what it ought to be, owing to his improper measures of the air's resistance; and therefore the conclusions derived by means of those rules must needs be very erroneous. Those of the very ingenious Mr. Simpson, contained in his Select Exercises, being partly founded on experiment, may bring out conclusions in some of the cases not very incorrect; while some of them, particularly those relating to the impetus and the time of flight, must be very wide of the truth. We must therefore refer the student, for more satisfaction, to our rules and examples before given in vol. 2 pa. 162 &c, especially for the circumstances of different ranges and elevations, &c, after having determined, as above, those for the greatest ranges, founded on the real measure of the resistances.

CHAPTER

CHAPTER XIV.

PROMISCUOUS PROBLEMS, AS EXERCISES IN MECHANICS, STATICS, DYNAMICS, HYDROSTATICS, HYDRAULICS, PROJECTILES, &c. &c.

PROBLEM I.

Let AB *and* AC *be two inclined planes, whose common altitude* AD *is given* = 64 *feet; and their lengths such, that a heavy body is* 2 *seconds of time longer in descending through* AB *than through* AC, *by the force of gravity; and if two balls, the one weighing* 3 *and the other* 2*lb, be connected by a thread and laid on the planes, the thread sliding freely over the vertex* A, *they will mutually sustain each other. Quere the lengths of the two planes?*

The lengths of planes of the same height being as the times of descent down them (art. 133 vol. 2), and also as the weights of bodies mutually sustaining each other on them (art. 122), therefore the times must be as the weights; hence as 1, the difference of the weights, is to 2 sec. the diff. of times, :: $\left\{\begin{matrix} 3 : 6 \text{ sec.} \\ 2 : 4 \text{ sec.} \end{matrix}\right\}$ the times of descending down the two planes. And as $\sqrt{16} : \sqrt{64} :: 1$ sec. : 2 sec. the time of descent down the perpendicular height (art. 70). Then, by the laws of descents (art. 132), as 2 sec. : 64 feet :: $\left\{\begin{matrix} 6 \text{ sec.} : 192 \\ 4 \text{ sec.} : 128 \end{matrix}\right\}$ feet, the lengths of the planes.

Note. In this solution we have considered 16 feet as the space freely descended by bodies in the 1st second of time, and 32 feet as the velocity acquired in that time, omitting the fractions $\frac{1}{12}$ and $\frac{1}{6}$, to render the numeral calculations simpler, as was done in the preceding chapter on projectiles, and as we shall do also in solving the following questions, wherever such numbers occur.

Another Solution by means of Algebra.

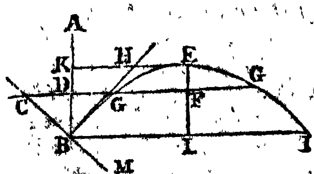
Put x = the time of descent down the less plane; then will $x + 2$ be that of the greater, by the question. Now the weights being as the lengths of the planes, and these again as the times, therefore as $2 : 3 :: x : x + 2$; hence

$2x +$

$2x + 4 = 3x$, and $x = 4$ sec. Then the lengths of the planes are found as in the last proportion of the former solution.

PROBLEM 2.

If an elastic ball fall from the height of 50 feet above the plane of the horizon, and impinge on the hard surface of a plane inclined to it in an angle of 15 degrees; it is required to find what part of the plane it must strike, so that after reflection, it may fall on the horizontal plane, at the greatest distance possible beyond the bottom of the inclined plane?



Here it is manifest that the ball must strike the oblique plane continued on a point somewhere below the horizontal plane; for otherwise there could be no maximum. Therefore let BC be the inclined plane, CDG the horizontal one, B the point on which the ball impinges after falling from the point A, BEGI the parabolic path, E its vertex, BH a tangent at B, being the direction in which the ball is reflected; and the other lines as are evident in the figure. Now, by the laws of reflection, the angle of incidence ABC, is equal to the angle of reflection HBM, and therefore this latter, as well as the former, is equal to the complement of the $\angle$ C the inclination of the two planes; but the part IBM is $= \angle$ C, therefore the angle of projection HBI is $=$ the comp. of double the $\angle$ C, and being the comp. of HBK, theref. $\angle$ HBK $= 2 \angle$ C. Now, put $a = 50 =$ AD the height above the horizontal line, $t =$ tang. $\angle$ DBC or 75° the complement of the plane's inclination, $\tau =$ tang. HBI or $\angle$ H $= 60^\circ$ the comp. of $2 \angle$ C, $s =$ sine of $2 \angle$ HBI $= 120^\circ$ the double elevation, or $=$ sine of $4 \angle$ C; also $x =$ AB the impetus or height fallen through. Then,

BI $= 4$KH $= 2sx$, by the projectiles prop. 21,

and $\left\{ \begin{array}{l} \text{BK} = \tau \times \text{KH} = \frac{1}{2}s\tau x \\ \text{CD} = t \times \text{BD} = t(x - a) \end{array} \right\}$ by trigonometry.

also, KD $=$ BK $-$ BD $= \frac{1}{2}s\tau x - x + a$, and KE $= \frac{1}{2}$BI $= sx$; then, by the parabola, $\sqrt{\text{BK}} : \sqrt{\text{DK}} :: \text{KE} : \text{FG} = \text{KE} \times \sqrt{\frac{\text{KD}}{\text{KB}}} = \sqrt{\frac{\tau s^2x^2 - 2sx^2 + 2asx}{\tau}} = \sqrt{[\frac{2s}{\tau}ax - (\frac{2s}{\tau} - s^2)x^2]} = 2b\sqrt{(ax - b^2x^2)}$, putting $b =$ sine of $2 \angle$ C $=$ sine of 30°. Hence CG $=$ CD $+$ DF $\pm$ FG $= tx - ta + sx \pm 2b\sqrt{(ax - b^2x^2)}$ a maximum, the fluxion of which made $= 0$, and the equation reduced, gives $x = \frac{a}{2b^2} \times (1 \pm \sqrt{\frac{n^2}{(n^2 + 4b^4)}}$, where $n = s$ $+ t$,

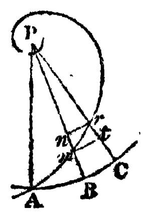
$+ t$, and the double sign $\pm$ answers to the two roots or values of x, or to the two points G, G, where the parabolic path cuts the horizontal line CG, the one in ascending and the other in descending.

Now, in the present case, when the $\angle$ C $= 15°$, $t =$ tang. $75° = 2 + \sqrt{3}$, $\tau =$ tan. $60° = \sqrt{3}$, $s =$ sin. $60° = \frac{1}{2}\sqrt{3}$, $b =$ sin. $30° = \frac{1}{2}$, $n = s + t = 2 + \frac{3}{2}\sqrt{3}$; then $\frac{a}{2b^2} = 2a = 100$, and $\frac{n^2}{n^2 + 4b^4} = \frac{n^2}{n^2 + \frac{1}{4}} = \frac{41 + 6\sqrt{3}}{52}$; theref. $x = \frac{a}{2b^2} \times (1 \pm \sqrt{\frac{n^2}{n^2 + 4b^4}})$ $= 100 \times (1 \pm \frac{1}{2}\sqrt{\frac{41 + 6\sqrt{3}}{13}}) = 100 \times (1 \pm \cdot 99414) = 199 \cdot 414$ or $\cdot 586$; but the former must be taken. Hence the body must strike the inclined plane at 149·414 feet below the horizontal line; and its path after reflection will cut the said line in two points; or it will touch it when $x = \frac{a}{bb}$. Hence also the greatest distance CG required is 826·9915 feet.

Corol. If it were required to find CG or $tx - ta + sx \pm 2b\sqrt{(ax - b^2x^2)} = g$ a given quantity, this equation would give the value of x by solving a quadratic.

PROBLEM 3.

Suppose a ship to sail from the Orkney Islands, in latitude 59° 3′ north, on a N. N. E. *course, at the rate of 10 miles an hour; it is required to determine how long it will be before she arrives at the pole, the distance she will have sailed, and the difference of longitude she will have made when she arrives there?*



Let ABC represent part of the equator; P the pole; AmrP a loxodromic or rhumb line, or the path of the ship continued to the equator; PB, PC, any two meridians indefinitely near each other; nr, or mt, the part of a parallel of latitude intercepted between them.

Put c for the cosine, and t for the tangent of the course, or angle nmr to the radius r; Am, any variable part of the rhumb from the equator, $= v$; the latitude Bm $= w$; its sine x, and cosine y; and AB, the dif. of longitude from A, $= z$. Then, since the elementary triangle mnr may be considered as a right-angled plane triangle, it is, as rad. $r : c =$ sin. $\angle\, mrn :: \dot{v} = mr : \dot{w} = mn :: v : w$; theref. $cv = rw$, or $v = \frac{rw}{c} = \frac{sw}{r}$, by putting s for the secant of the $\angle\, nmr$ the ship's course. In like manner,

ner, if w be any other latitude, and v its corresponding length of the rhumb; then $v = \frac{rw}{c}$; and hence $V - v = r \times \frac{W - w}{c}$, or $D = \frac{rd}{c}$, by putting $D = V - v$ the distance, and $d = W - w$ the dif. of latitude; which is the common rule.

The same is evident without fluxions: for since the $\angle\, mrn$ is the same in whatever point of the path AmrP the point m is taken, each indefinitely small particle of AmrP, must be to the corresponding indefinitely small part of Bm, in the constant ratio of radius to the cosine of the course; and therefore the whole lines, or any corresponding parts of them, must be in the same ratio also, as above determined. In the same manner it is proved that radius : sine of the course :: distance : the departure.

Again, as radius $r : t =$ tang. $nmr :: \dot{w} = mn : nr$ or mt, and as $r : y ::$ PB : Pm :: $\dot{z} =$ BC : mt; hence, as the extremes of these proportions are the same, the rectangles of the means must be equal, viz, $y\dot{z} = t\dot{w} = \frac{tr\dot{x}}{y}$ because $\dot{w} = \frac{r\dot{x}}{y}$ by the property of the circle; theref. $\dot{z} = \frac{tr\dot{x}}{y^2} = \frac{tr\dot{x}}{r^2 - x^2}$; the general fluents of these are $z = t \times$ hyp. log. $\sqrt{\frac{r+x}{r-x}} + c$; which corrected by supposing $z = 0$ when $x = a$, are $z = t \times$ (hyp. log. $\sqrt{\frac{r+x}{r-x}}$ − hyp. log. $\sqrt{\frac{r+a}{r-a}}$); but $r \times$ (hyp. log. $\sqrt{\frac{r+x}{r-x}}$ − hyp. log. $\sqrt{\frac{r+a}{r-a}}$) is the meridional parts of the dif. of the latitudes whose sines are x and a, which call b; then is $z = \frac{tb}{r}$, the same as it is by Mercator's sailing.

Further, putting $m = 2{\cdot}71828$ the number whose hyp. log. is 1, and $n = \frac{2z}{t}$; then, when z begins at A, $m^n = \frac{r+x}{r-x}$, and theref. $x = r \times \frac{m^n - 1}{m^n + 1} = r - \frac{2r}{m^n + 1}$: hence it appears that as m^n, or rather n or z increases (since m is constant), that x approximates to an equality with r, because $\frac{2r}{m^n + 1}$ decreases or converges to 0, which is its limit; consequently r is the limit or ultimate value of x: but when $x = r$, the ship will be at the pole; theref. the pole must be the limit, or evanescent state, of the rhumb or course: so that the ship may be said to arrive at the pole after making an infinite number of revolutions round it; for the above expression $\frac{2r}{m^n + 1}$ vanishes when n, and consequently z, is infinite, in which case x is $= r$.

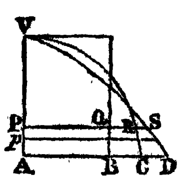
 Now,

Now, from the equation $D = \frac{rd}{c} = \frac{sd}{r}$, it is found, that, when $d = 30° \ 57'$ the comp. of the given lat. $59° \ 3'$, and $c =$ sine of $67° \ 30'$ the comp. of the course, D will be $= 2010$ geographical miles, the required ultimate distance; which, at the rate of 10 miles an hour, will be passed over in 201 hours or $8\frac{3}{8}$ days. The dif. of long. is shown above to be infinite. When the ship has made one revolution, she will be but about a yard from the pole, considering her as a point.

When the ship has arrived infinitely near the pole, she will go round in the manner of a top, with an infinite velocity; which at once accounts for this paradox, viz, that though she make an infinite number of revolutions round the pole, yet her distance run will have an ultimate and definite value, as above determined: for it is evident that however great the number of revolutions of a top may be, the space passed over by its pivot or bottom point, while it continues on or nearly on the same point, must be infinitely small, or less than a certain assignable quantity.

PROBLEM 4.

A current of water is discharged by three equal openings or sluices, in the following shapes: the first a rectangle, the second a semicircle, and the third a parabola, having their altitudes equal, and their bases in the same horizontal line, and the water level with the tops of the arches: on this supposition it is required to show what may be the proportion of the quantities discharged by these sluices.



Let VB be half the parallelogram, AVC half the semicircle, and AVD half the parabola, that is, the halves of the respective sluices or gates. Put $a =$ AV the common altitude, and $c = \cdot 7854$: then is ca^2 the area of each of the figures; also $ca =$ AB, $a =$ AC, and $\frac{3}{2}ca =$ AD; also put $x =$ VP any variable depth, and $\dot{x} =$ Pp. Then, the water discharged, at any depth x, being as the velocity and aperture, and the velocity being in all the figures as $\sqrt{x}$, therefore $\dot{x}\sqrt{x} \times$ PQ, and $\dot{x}\sqrt{x} \times$ PR, and $\dot{x}\sqrt{x} \times$ PS, or $cax^{\frac{1}{2}}\dot{x}$, and $x\dot{x}\sqrt{(2a-x)}$, and $\frac{3}{2}c\sqrt{a} \times x\dot{x}$, are proportional to the fluxions of the quantity of water discharged by the said figures or sluices respectively; the correct fluents of which, when $x = a$, are $\frac{2}{3}ca^{\frac{5}{2}}$, and $\frac{2}{15}a^{\frac{5}{2}}(8\sqrt{2} - 7)$, and $\frac{3}{4}ca^{\frac{5}{2}}$, the 2d fluent being found by art. 12 pa. 225 of this vol. Hence the quantities of

of water discharged by the rectangle, the semicircle, and the parabola, are respectively as $\frac{2}{3}c$, and $\frac{2}{15}(8\sqrt{2}-7)$, and $\frac{4}{5}c$, or as 1, and $\frac{2}{5c}(8\sqrt{2}-7)$, and $\frac{6}{5}$, or as 1, and 1·09847, and $1\frac{1}{5}$.

PROBLEM 5.

The initial velocity of a 24lb ball of cast iron, which is projected in a direction perpendicular to the horizon, being supposed 1200 feet per second; and that the resistance of the medium is constantly as the square of the velocity, and everywhere of the same density: required the time of flight, and the height to which it will ascend.

Answer. By problems 5 and 6, of the last chapter, the ascent will be found = 5337 feet, and the time of the ascent 28 seconds.

PROBLEM 6.

To determine the same as in the last question, supposing the density of the atmosphere to decrease in ascending after the usual way?

Ans. By probs. 7 and 8, the height will be 5614 feet, and the time 34 seconds.

PROBLEM 7.

It is required to find the diameter of a circular parachute, by means of which a man of 150lb weight may descend on the earth, from a balloon at a height in the air, with the velocity of only 10 feet in a second of time, being the velocity acquired by a body freely descending through a space of only 1 foot $6\frac{3}{4}$ inches, or of a man jumping down from a height of $18\frac{3}{4}$ inches: the parachute being made of such materials and thickness, that a circle of it of 50 feet diameter, weighs only 150lb, and so in proportion more or less according to the area of the circle.

If a falling body descend with a uniform velocity, it must necessarily meet with a resistance, from the medium it descends in, equal to the whole weight that descends. Let x denote the diameter of the parachute, and $a = \cdot7854$; then ax^2 will be its area, and as $50^2 : x^2 :: 150 : \frac{3}{50}x^2$ the weight of the same, to which adding 150lb, the man's weight, the sum $\frac{3}{50}x^2 + 150$ will be the whole descending weight. Again, in the table of resistances at pa. 375 near the end of vol. 2, we find that a circle of $\frac{1}{2}$ of a square foot area, moving with 10 feet velocity, meets with a resistance of ·57 ounces = ·0475lb; and the resistances, with the same velocity, being as

as the surfaces, therefore as $\frac{2}{9}$: $\cdot 0475$ :: ax^2 : $\cdot 21375ax^2 = \cdot 16788x^2$ the resistance of the air to the parachute, to which the descending weight must be equal; that is, $\cdot 16788x^2 = \frac{3}{50}x^2 + 150$; hence $\cdot 10788x^2 = 150$, or $x^2 = 1390 \cdot 5$, and hence $x = 37\frac{2}{7}$ feet, the diameter of the parachute required.

PROBLEM 8.

To determine the effects of Pile-Engines.

The form of the pile-engine, as used by the ancients, is not known. Many have been invented and described by the moderns. Among all these, that appears to be the best which was invented by Vauloue, as described by Desaguliers, and was used at piling the foundations at building Westminster Bridge. Its chief properties are, that the ram or weight be raised with the least expence of force, or with the fewest men; that it fall freely from its greatest height; and that, having fallen, it is presently laid hold of by the forceps, and so raised up to its height again. By which means, in the shortest time, and with the fewest men, or the least force, the most piles can be driven to the greatest depth.

Belidor has given some theory as to the effect of the pile-engine, but it appears to be founded on an erroneous principle: he deduces it from the laws of the collision of bodies. But who does not perceive that the rules of collision suppose a free motion and a non-resisting medium? It cannot therefore be applied in the present case, where a very great resistance is opposed to the pile by the ground. We shall therefore here endeavour to explain another theory of this machine.

Since the percussion of the weight acts on the pile during the whole time the pile is penetrating and sinking in the earth, by each blow of the ram, during which time its whole force is spent; it is manifest that the effect of the blow is of that nature, which requires the force of the blow to be estimated by the square of the velocity. But the square of the velocity acquired by the fall of the ram, is as the height it falls from; therefore the force of any blow will be as the height fallen through. But it is also more or less in proportion to the weight of the ram; consequently the effect or force of each blow must be directly in the compound ratio of both, viz, as aw, where w denotes the weight, and a the altitude it falls from; or it will be simply as the altitude a, when the weight w is constant.

Again, the force of the blow is opposed by the mass of the pile, and by the consistence of the earth penetrated by the point

point of the pile, and also by the friction of the earth against the surface or sides of the pile that have penetrated below the surface. Consequently the effect of the blow, or the depth penetrated by the pile, will be inversely in the compound ratio of these three, viz, inversely as mtf, where m denotes the mass of the pile, t the tenacity or cohesion of the earth, and f the friction of the surface penetrated in the earth. But, in the same soil and with the same pile, m and t are both constant, in which case the depth of penetration will be inversely only as f the friction. On all accounts then the penetration will be as $\frac{aw}{mtf}$, or simply as $\frac{a}{f}$ only, for the same weight and pile and soil.

To determine the depth sunk by the pile at each stroke of the ram.

After a few strokes, so as to give the pile a little hold in the ground, to make it stand firmly, the blows of the ram may be considered as commencing, and causing the pile to sink a little at every stroke, by which small successive sinkings of the pile, the space the ram falls through will be successively increased by these small accessions, and the force of the successive blows proportionally increased. But these, on the other hand, are resisted and opposed by the friction of the part of the pile which has been sunk before, and which also sinks at each stroke; and as the quantities of these rubbing surfaces increase in a greater ratio to each other, than the heights fallen through, that is, the resisting forces increasing faster than the impelling forces, it is manifest that the depths successively sunk by the blows must gradually decrease by little and little every time; which is also found to be quite conformable to experience. Thus then the successive sinkings will proceed gradually diminishing, till they become so small as to be almost imperceptible.

Now it was found above that $\frac{a}{f}$ is as the penetration by any blow of the ram, by the same pile in the same soil, that is, as the height fallen directly, and as the resistance or friction in the earth inversely. Let A denote any other and greater height, by an after stroke, and F its friction; also P the penetration by the former blow, and p that by the latter, which must be the smaller: then, by the foregoing principle, $\frac{a}{f} : \frac{A}{F} :: P : p$; hence $a : A :: fP : Fp$, which is a general theorem.

But

But now, with respect to the quantity of friction from any blow, though it be not known from experiment that the friction is exactly proportional to the rubbing surface, there is great reason to believe that it must be at least very nearly so: there is also equal reason to conclude that the effect or resistance from that rubbing surface must be nearly or exactly as the length of space it moves over, that is by the penetration of the pile by any blow. Now, if d denote the depth of the pile in the ground before any new blow is struck by the ram, and b the depth or penetration produced by the blow, then the length of the rubbing surface will be $d + \frac{1}{2}b$; for, the length of the rubbing surface is only d at the beginning of the motion, and it is $d + b$ at the end of it, the medium of the two, or $d + \frac{1}{2}b$, is therefore the due length of the surface, and the space or depth it moves over is b; therefore the whole resistance from the friction is $(d + \frac{1}{2}b)b$. If D then denote any other depth of the pile in the earth, and b' the next penetration, then $(D + \frac{1}{2}b')b'$ will be its friction. Substituting now b for P, and b' for p, also $d + \frac{1}{2}b$ for f, and $D + \frac{1}{2}b'$ for F, in the general theorem $a : A :: fP : Fp$, it becomes $a : A :: (d + \frac{1}{2}b)b : (D + \frac{1}{2}b')b'$, for the general relation between the heights fallen and the resistance and penetration.

This theorem will very conveniently give the series of effects, or successive sinkings of the piles, by the blows of the ram. Thus, after the pile has been properly fixed, or indeed driven to any depth in the earth, denoted by d, then to give a blow, the ram falls from the height $a + d$, and thereby sinks the pile the space b suppose: hence, for the next stroke, the fall will be $a + d + b = A$ in the theorem above, and $D + \frac{1}{2}b' = d + b + \frac{1}{2}b'$, the next penetration or sinking being b'; theref. $a + d : a + d + b :: (d + \frac{1}{2}b)b : (d + b + \frac{1}{2}b')b'$, a proportion which gives the quadratic equa. $b'^2 + 2b'(d+b) = \frac{a+d+b}{a+d} \times (2d + b)b$, the root of which is $b' = -(d + b) +$

$$\sqrt{[(d + b)^2 + \frac{a+d+b}{a+d} \times (2d + b)b]} = \frac{a+d+b}{a+d} \times \frac{d + \frac{1}{2}b}{d+b}b$$

nearly, or indeed $= \frac{d+\frac{1}{2}b}{d+b}b$ nearly, because b is small in comparison with $a + d$.

Now, for an example in numbers, suppose $a = 5$ feet $= 60$ inches, $d = 10$, $b = 3$, that is $a = 60$ the height of the ram above the top of the pile before this enters the ground; $d = 10$, after being fixed in the ground; and $b = 3$ the sinking by the next blow: then $\frac{d+\frac{1}{2}b}{d+b}b = \frac{11 \cdot 5}{13} \times 3 = 2 \cdot 65 = b'$, the

the 2d stroke. Next, substituting $d+b$ for d, and b' for b, the same theorem gives 2·48 for the next sinking, or the next value of b'. And so on continually, by which means the series of the successive corresponding values of the letters will be as in the margin, the last column showing the several successive sinkings of the pile by the repeated strokes of the ram.

Specimen of the Series of the successive values of d, b, b'.

d	b	b'
10	3	2·65
13	2·65	2·48
15·65	2·49	2·32
18·14	2·32	2·19
20·46	2·19	2·08
&c.		

Scholium. Thus then it appears, that the effect of any operation of pile-driving may be determined. It is manifest also that the greater a is, or the higher the top of the machine is where the ram falls from, above the top of the pile at first, the greater will be every stroke of the ram, and consequently the fewer the strokes necessary to drive the pile to the requisite depth. But then every stroke will take a longer time, as the ram will be both longer in falling and longer in raising: so that it may be a question whether, on the whole, the business may be effected in the less time by a greater height of the machine, or whether there be any limit to the height, so as to produce the greatest effect in a given time.

To answer this question, let x denote the indeterminate height from which any weight w is to fall, z the time of raising it after a fall, which time is supposed to be as the height x to which it is raised, also m the given time of producing a proposed effect; then $\frac{1}{4}\sqrt{x} =$ the time of the weight falling; therefore $\frac{1}{4}\sqrt{x}+z =$ the whole time of one stroke; conseq. $\frac{m}{\frac{1}{4}\sqrt{x}+z}$ or $\frac{4m}{\sqrt{x}+4z}$ is the number of strokes made in the given time m, and hence $\frac{4mxw}{\sqrt{x}+4z} =$ the whole force or effect in the time m. Now this effect or fraction increases continually as x increases, because the numerator increases faster than the denominator, since the former increases as x, while in the latter though the one term z increases as x, yet the other term $\sqrt{x}$ only increases as the root of x. So that, on the whole, it appears that the effect, in any given time, increases more and more as the height is increased.

PROBLEM 9.

To determine how far a man, who pushes with the force of 100lb, can force a sponge into a piece of ordnance, whose diameter is 5 inches, and length 10 feet, when the barometer stands at 30 inches; the vent, or touch-hole, being stopped, and the sponge having no windage, that is, fitting the bore quite close?

A column of quicksilver 30 inches high, and 5 in diameter, is $5^2 \times 30 \times \cdot 7854 = 589\cdot 05$ inches; which, at 8·102 oz each inch, weighs 4772·48 oz or 298·28 lb, which is the pressure of the atmosphere alone, being equal to the elasticity of the air in its natural state; to this adding the 100 lb, gives 398·28 lb, the whole external pressure. Then, as the spaces which a quantity of air possesses, under different pressures, are in the reciprocal ratio of those pressures, it will be, as 398·28 : 298·28 : : 10 feet or 120 inches : 90 inches nearly, the space occupied by the air; theref. $120 - 90 = 30$ inches, is the distance sought.

PROBLEM 10.

To assign the Cause of the Deflection of Military Projectiles.

It having been surmized that, in the practice of artillery, the deflection of the shot in its flight, to the right or left, from the line or direction the gun is laid in, chiefly arises from the motion of the gun during the time the shot is passing out of the piece; it is required to determine what space an 18 pounder will recoil or fly back, while the shot is passing out of the gun; supposing its weight to be 4800 lb, that of the carriage 2400 lb, the quantity of powder 8 lb, the length of the cylinder 108 inches, that of the charge 13 inches, and the diameter of the bore 5·13 inches; supposing also that the resistance from the friction between the platform and carriage is equal to 3600 lb?

It is well known that confined gunpowder, when fired, immediately changes in a great measure into an elastic air, which endeavours to expand in all directions. Now, in the question, the action of this fluid is exerted equally on the bottom of the bore of the gun and on the ball, during the passage of the latter through the cylinder; the two bodies therefore move in opposite directions, with velocities which are at all times in the inverse ratio of the quantities of matter moved. Now let x be the space through which the gun recoils; then, as the charge occupies 13 inches of the barrel, and the semidiameter of the barrel is 2·565, the space moved

through

through by the ball when it quits the piece, is $108 - 13 - 2\cdot565 - x = 92\cdot435 - x$: and as the elastic fluid expands in both directions, the quantity which advances towards the muzzle, is to that which retreats from it, as $92\cdot435 - x$ to x: conseq. $\frac{8x}{92\cdot435}$ and $\frac{92\cdot435 - x}{92\cdot435} \times 8$ are the quantities of the powder which move, the former with the gun, and the latter with the ball; besides these, the weight of ball that moves forwards being 18lb, and of the weights and resistance backwards $4800 + 2400 + 3600 = 10800$lb, hence the whole weights moved in the two directions are $10800 + \frac{8x}{92\cdot435}$ and $18 + \frac{92\cdot435 - x}{92\cdot435} \times 8$, or $\frac{998298 + 8x}{92\cdot435}$ and $\frac{2403\cdot31 - 8x}{92\cdot435}$, or as the numerators of these only. But when the time and moving force are given, or the same, then the spaces are inversely as the quantities of matter; therefore $x : 92\cdot435 - x :: 2403\cdot31 - 8x : 998298 + 8x$, or by composition, $x : 92\cdot435 :: 2403\cdot31 - 8x : 1000701\cdot31$, and by div. $x : 1 :: 2403\cdot31 - 8x : 10826$, theref. $10826x = 2403\cdot31 - 8x$, or $10834x = 2403\cdot31$, and hence $x = \cdot2218$ inch $= \frac{2}{9}$ of an inch nearly, or the recoil of the gun is less than a quarter of an inch.

Hence it may be concluded, that so small a recoil, straight backwards, can have no effect in causing the ball to deviate from the pointed line of direction: and that it is very probable we are to seek for the cause of this effect in the ball striking or rubbing against the sides of the bore, in its passage through it, especially near the exit at the muzzle; by which it must happen, that if the ball strike against the right side, the ball will deviate to the left; if it strike on the left side, it must deviate to the right; if it strike against the under side, it must throw the ball upwards, and make it to range farther; but if it strike against the upper side, it must beat the ball downwards, and cause a shorter range: all which irregularities are found to take place, especially in guns that have much windage, or which have the balls too small for the bore.

PROBLEM 11.

A ball of lead of 4 inches diameter, is dropped from the top of a tower, of 65 yards high, and falls into a cistern full of water at the bottom of the tower, of $20\frac{1}{4}$ yards deep: it is required to determine the times of falling, both to the surface and to the bottom of the water.

The fall in air is 195 feet, and in water $60\frac{3}{4}$ feet. By the common rules of descent, as $\sqrt{16} : \sqrt{195} :: 1'' : \frac{1}{4}\sqrt{195} =$ 3·49

3·49 seconds, the time of descending in air. And as $\sqrt{16}$: $\sqrt{195}$ :: 32 : $8\sqrt{195} = 111·71$ feet, the velocity at the end of that time, or with which the ball enters the water.

Again, by prob. 22 of vol. 2, art. 2, the space $s = \frac{1}{2b} \times$ hyp. log. of $\frac{a-e^2}{a-v^2}$, or rather $\frac{1}{2b} \times$ hyp. log. of $\frac{e^2-a}{v^2-a}$ (the velocity being decreasing, and e^2 greater than a) $= \frac{m}{2b} \times$ com. log. of $\frac{e^2-a}{v^2-a}$, where $N = 11325$ the density of lead, $n = 1000$ that of water, $a = \frac{256d(N-n)}{3n}$, $b = \frac{3n}{8dN}$, $e = 111·71$ the velocity at entering the water, and v the velocity at any time afterwards, also d the diameter of the ball = 4 inches, and m = 2·302585 the hyp. log. of 10.

Hence then $N = 11325$, $n = 1000$, $N - n = 10325$, $d = \frac{4}{12} = \frac{1}{3}$; then $a = \frac{256d(N-n)}{3n} = \frac{256 \,.\, 10325}{9000} = 293\frac{1}{2}$, and $b = \frac{3n}{8dN} = \frac{9n}{8N} = \frac{9000}{90600} = \frac{15}{151} = \frac{1}{10}$ nearly. Also $e = 111·71$; therefore $s = 60\frac{3}{4} = \frac{m}{2b} \times$ log. of $\frac{e^2-a}{v^2-a} = 5m \times$ log. $\frac{e^2-a}{v^2-a}$. This theorem will give s when v is given, and by reverting, it will give v in terms of s in the following manner.

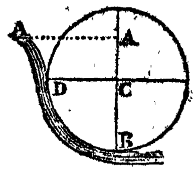
Dividing by $5m$, gives $\frac{s}{5m}$ = log. of $\frac{e^2-a}{v^2-a} = ns$, by putting $n = \frac{1}{5m}$; therefore, the natural number is $10^{ns} = \frac{e^2-a}{v^2-a}$; hence $v^2 - a = \frac{e^2-a}{10^{ns}}$, and $v = \sqrt{(a + \frac{e^2-a}{10^{ns}})}$, which, by substituting the numbers above mentioned for the letters, gives $v = 17·134$ for the last velocity, when the space $s = 60\frac{3}{4}$, or when the ball arrives at the bottom of the water.

But now to find the time of passing through the water, putting t = any time in motion, and s and v the corresponding space and velocity, the general theorem for variable forces gives $\dot{t} = \frac{\dot{s}}{v}$. But the above general value of s being $\frac{1}{2b} \times$ hyp. log. $\frac{e^2-a}{v^2-a}$ or $5 \times$ hyp. log. $\frac{e^2-a}{v^2-a}$, therefore its fluxion $\dot{s} = \frac{-10v\dot{v}}{v^2-a}$, conseq. $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{-10\dot{v}}{v^2-a}$, the correct fluent of which is $\frac{5}{\sqrt{a}} \times$ hyp. log. $(\frac{e-\sqrt{a}}{e+\sqrt{a}} \times \frac{v+\sqrt{a}}{v-\sqrt{a}}) = t$ the time, which when $v = 17·134$, or $s = 60\frac{3}{4}$, gives 2·6542 seconds, for the time of descent through the water.

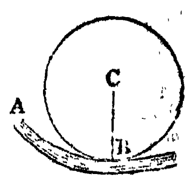
PROBLEM

PROBLEM 12.

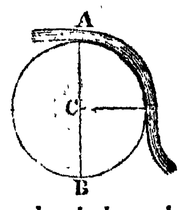
Required to determine what must be the diameter of a water-wheel, so as to receive the greatest effect from a stream of water of 12 feet fall?



In the case of an undershot wheel, put the height of the water AB $= 12$ feet $= a$, and the radius BC or CD of the wheel $= x$, the water falling perpendicularly on the extremity of the radius CD at D. Then AC or AD $= cx$, and the velocity due to this height, or with which the water strikes the wheel at D, will be as $\sqrt{(a-x)}$, and the effect on the wheel being as the velocity and as the length of the lever CD, will be denoted by $x\sqrt{(a-x)}$ or $\sqrt{(ax^2-x^3)}$, which therefore must be a maximum, or its square ax^2-x^3 a maximum. In fluxions, $2ax\dot{x}-3x^2\dot{x}=0$; and hence $x=\frac{2}{3}a=8$ feet the radius.



But if the water be considered as conducted so as to strike on the bottom of the wheel, as in the annexed figure, it will then strike the wheel with its greatest velocity, and there can be no limit to the size of the wheel, since the greater the radius or lever BC, the greater will be the effect.



In the case of an overshot wheel, $a-2x$ will be the fall of water, $\sqrt{(a-2x)}$ as the velocity, and $x\sqrt{(a-2x)}$ or $\sqrt{(ax^2-2x^3)}$ the effect, then ax^2-2x^3 is a maximum, and $2ax\dot{x}-6x^2\dot{x}=0$; hence $x=\frac{1}{3}a=4$ feet is the radius of the wheel.

But all these calculations are to be considered as independent of the resistance of the wheel, and of the weight of the water in the buckets of it.

PROBLEM 13.

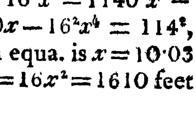
What angle must a projectile make with the plane of the horizon, discharged with a given velocity v, so as to describe in its flight a parabola including the greatest area possible?

By the set of theorems in art. 92 pa. 156 vol. 2, for any proposed angle, there can be assigned expressions for the horizontal range and the greatest height the projectile rises to, that is the base and axis of the parabolic trajectory. Thus, putting s and c for the sine and cosine of the angle of elevation;

tion; then, by the first line of those theorems, the velocity being v, the horizontal range R is $= \frac{1}{16}scv^2$; and, by the 4th or last line of theorems, the greatest height H is $= \frac{1}{64}s^2v^2$. But, by the parabola, $\frac{2}{3}$ of the product of the base or range and the height is the area, which is now required to be the greatest possible. Therefore R × H $= \frac{1}{16}scv^2 \times \frac{1}{64}s^2v^2$ must be a maximum, or, rejecting the constant factors, s^3c a maximum. But the cosine c, of the angle whose sine is s, is $\sqrt{(1-s^2)}$; therefore $s^3c = s^3\sqrt{(1-s^2)} = \sqrt{(s^6-s^8)}$ is the maximum, or its square $s^6 - s^8$ a maximum. In fluxions $6s^5\dot{s} - 8s^7\dot{s} = 0 = 3 - 4s^2$; hence $4s^2 = 3$, or $s^2 = \frac{3}{4}$, and $s = \frac{1}{2}\sqrt{3} = \cdot8660254$, the sine of $60°$, which is the angle of elevation to produce a parabolic trajectory of the greatest area.

PROBLEM 14.

Suppose a cannon were discharged at the point A; *it is required to determine how high in the air the point* C *must be raised above the horizontal line* AB, *so that a person at* C *letting fall a leaden bullet at the moment of the cannon's explosion, it may arrive at* B *at the same instant as he hears the report of the cannon, but not till* $\frac{1}{10}$*th of a second after the sound arrives at* B: *supposing the velocity of sound to be* 1140 *feet per second, and that the bullet falls freely without any resistance from the air?*

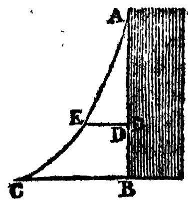


Let x denote the time in which the sound passes to C; then will $x - \frac{1}{10}$ be the time in passing to B, and x the time also the bullet is falling through CB. Then, by uniform motion, $1140x =$ AC, and $1140x - 114 =$ AB, also by descents of gravity, $1^2 : x^2 :: 16 : 16x^2 =$ BC. Then, by right-angled triangles, $AC^2 - BC^2 = AB^2$, that is $1140^2x^2 - 16^2x^4 = 1140^2x^2 - 224 \times 1140x + 114^2$, hence $224 \times 1140x - 16^2x^4 = 114^2$, or $1015{\cdot}3x - x^4 = 50{\cdot}77$, the root of which equa. is $x = 10{\cdot}03$ seconds, or nearly 10 seconds; conseq. $BC = 16x^2 = 1610$ feet nearly, the height required.

PROBLEM 15.

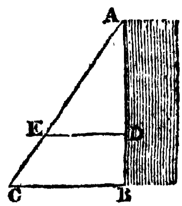
Required the quantity, in cubic feet, of light earth, necessary to form a bank on the side of a canal, which will just support a pressure of water 5 feet deep, and 300 feet long. And what will the carriage of the earth cost, at the rate of 1 shilling per ton?

This



This question may be considered as relating either to water sustained by a solid wall, or by a bank of loose earth. In the former case, let ABC denote the wall, sustaining the pressure of the water behind it. Put the whole altitude AB $= a$, the base BC or thickness at bottom $= b$, any variable depth AD $= x$, and the thickness there DE $= y$. Now the effect which any number of particles of the fluid pressing at D have to break the wall at B, or to overturn it there, is as the number of particles AD or x, and as the lever BD $= a - x$; therefore the fluxion of the effect of all the forces is $(a-x)x\dot{x} = ax\dot{x} - x^2\dot{x}$, the fluent of which is $\frac{1}{2}ax^2 - \frac{1}{3}x^3$, which, when $x = a$, is $\frac{1}{6}a^3$ for the whole effect to break or overturn the wall at B; and the effects of the pressure to break at B and D will be as AB^3 and AD^3. But the strength of the wall at D, to resist the fracture there, like the lateral strength of timber, is as the square of the thickness, DE^2. Hence the curve line AEC, bounding the back of the wall, so as to be every where equally strong, is of such a nature, that x^3 is always proportional to y^2, or y as $x^{\frac{3}{2}}$, and is therefore what is called the semicubical parabola.

Now, to find the area ABC, or content of the wall bounded by this convex curve, the general fluxion of the area $y\dot{x}$ becomes $x^{\frac{3}{2}}\dot{x}$, the fluent of which is $\frac{2}{5}x^{\frac{5}{2}} = \frac{2}{5}xx^{\frac{3}{2}} = \frac{2}{5}xy$, that is $\frac{2}{5}$ of the rectangle AB $\times$ BC; and is therefore less than the triangle ABC, of the same base and height, in the proportion of $\frac{2}{5}$ to $\frac{1}{2}$, or of 4 to 5.



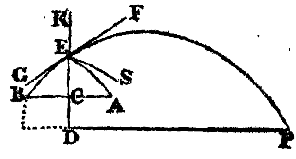
But in the case of a bank of made earth, it would not stand with that concave form of outside, if it were necessary, but would dispose itself in a straight line AC, forming a triangular bank ABC. And even if this were not the case naturally, it would be proper to make it such by art; because now neither is the bank to be broken as with the effect of the lever, or overturned about the pivot or point C, nor does it resist the fracture by the effect of a lever, as before; but, on the contrary, every point is attempted to be pushed horizontally outwards, by the horizontal pressure of the water, and it is resisted by the weight or resistance of the earth at any part, DE. Here then, by hydrostatics, the pressure of the water against any point D, is as the depth AD; and, in the triangle of earth ADE, the resisting quantity in DE is as DE, which

which is also proportional to AD by similar triangles. So that, at every point D in the depth, the pressure of the water and the resistance of the soil, by means of this triangular form, increase in the same proportion, and the water and the earth will everywhere mutually balance each other, if at any one point, as B, the thickness BC of earth be taken such as to balance the pressure of the water at B, and then the straight line AC be drawn, to determine the outer shape of the earth. All the earth that is afterwards placed against the side AC, for a convenient breadth at top for a walking path, &c, will also give the whole a sufficient security.

But now to adapt these principles to the numeral calculation proposed in the question; the pressure of water against the point B being denoted by the side AB = 5 feet, and the weight of water being to earth as 1000 to 1984, therefore as 1984 : 1000 : : 5 : 2·52 = BC, the thickness of earth which will just balance the pressure of the water there; hence the area of the triangle ABC = ½AB × BC = 2½ × 2·52 = 6·3; this mult. by the length 300, gives 1890 cubic feet for the quantity of earth in the bank; and this multiplied by 1984 ounces, the weight of 1 cubic foot, gives, for the weight of it, 3749760 ounces = 234360lbs = 104·625 tons; the expense of which, at 1 shilling the ton, is 5l. 4s. 7½d.

PROBLEM 16.

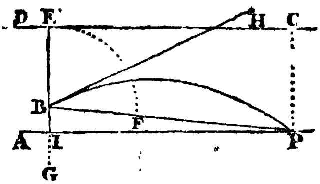
A person standing at the distance of 10 *feet from the bottom of a wall, which is supposed perfectly smooth and hard, desires to know in what direction he must throw an elastic ball against it, with a velocity of* 80 *feet per second, so that, after reflection from the wall, it may fall at the greatest distance possible from the bottom, on the horizontal plane, which is* 2½ *feet below the hand discharging the ball?*



In the annexed figure let DR be the wall against which the ball is thrown, from the point A, in such a direction, that it shall describe the parabolic curve AE before striking the wall, and afterwards be so reflected as to describe the curve EP. Now if ES be the tangent at the point E, to the curve AE described before the reflection, and EF the tangent at the same point to the curve which the ball will describe after reflection, then will the angle REF be = CES; and if the curve PE be produced, so as to have GF for its tangent, it will meet AC produced in B, making BC = AC, and the curve AE will be

similar

similar and equal to the portion BE of the parabola BEP, but turned the contrary way. Conceiving either the two curves AE and EP, or the continued curve BEP, to be described by a projectile in its motion, it is manifest that, whether the greater portion of the curve be described before or after the ball reaches the wall DR, will depend on its initial velocity, and on the distance AC or BC, and on the angle of projection. The problem then is now reduced to this, viz, To find the angle at which a ball shall be projected from B, with a given impetus, so that the distance DP, at which it falls, from the given point D, on the plane DP, parallel to the horizon, shall be a maximum.



Now this problem may be constructed in the following manner: From any point E in the horizontal line DC, let fall the indefinite perp. EG, on which set off EB = the impetus corresponding to the given velocity, and BI = 2½ the distance of the horizontal plane below the point of projection; also, through I draw AP parallel to DC. From the point B set off BP = BE + EI, and bisect the angle EBP by the line BH: then will BH be the required direction of the ball, and IP the maximum range on the plane AP.

For, since the ball moves from the point B, with the velocity acquired by falling through EB, it is manifest, from p. 156 vol. 2, that DC is the directrix of the parabola described by the ball. And since both B and P are points in the curve, each of them must, from the nature of the parabola, be as far from the focus as it is from the directrix; therefore B and P will be the greatest distance from each other when the focus F is directly between them, that is, when BP = BE + CP. And when BP is a maximum, since BI is constant, it is obvious that IP is a maximum too. Also, the angle FBH being = EBH, the line BH is a tangent to the parabola at the point B, and consequently it is the direction necessary to give the range IP.

Cor. 1. When B coincides with I, IP will be = BP = BE + EI = 2EI, and the angle EBH will be 45°: as is also manifest from the common modes of investigation.

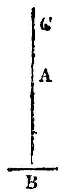
Cor. 2. When the impetus corresponding to the initial velocity of the ball is very great compared with AC or BC (fig. 1), then the part AE of the curve will very nearly coincide with its tangent, and the direction and velocity at A may be accounted the same as those at E without any sensible error.

error. In this case too the impetus BE (fig. 2) will be very great compared with BI, and consequently, B and I nearly coinciding, the angle EBH will differ but little from 45°.

Calcul. From the foregoing construction the calculation will be very easy. Thus, the first velocity being 80 feet $= v$, then (vol. 2 pa. 156) $\frac{v^2}{4g} = \frac{80 \times 80}{64\frac{1}{3}} = 99\cdot48186 =$ BE the impetus; hence EI $=$ FP $= 101\cdot98186$, and BP $=$ BE $+$ EI $= 201\cdot46372$. Now, in the right-angled triangle BIP, the sides BI and BP are known, hence IP $= 201\cdot4482$, and the angle IBP $= 89° \; 17' \; 20''$: half the suppl. of this angle is $45° \; 21' \; 20''$ $=$ EBH. And, in fig. 1, IP $-$ ID $= 201\cdot4482 - 10 = 191\cdot4482 =$ DP, the distance the ball falls from the wall after reflection.

PROBLEM 17.

From what height above the given point A *must an elastic ball be suffered to descend freely by gravity, so that, after striking the hard plane at* B, *it may be reflected back again to the point* A, *in the least time possible from the instant of dropping it?*



Let C be the point required; and put AC $= x$, and AB $= a$; then $\frac{1}{4}\sqrt{CB} = \frac{1}{4}\sqrt{(a+x)}$ is the time in CB, and $\frac{1}{4}\sqrt{CA} = \frac{1}{4}\sqrt{x}$ is the time in CA: therefore $\frac{1}{4}\sqrt{(a+x)} - \frac{1}{4}\sqrt{x}$ is the time down AB, or the time of rising from B to A again: hence the whole time of falling through CB and returning to A, is $\frac{1}{2}\sqrt{(a+x)} - \frac{1}{4}\sqrt{x}$, which must be a min. or $2\sqrt{(a+x)} - \sqrt{x}$ a minimum, in fluxions $\frac{\dot{x}}{\sqrt{(a+x)}} - \frac{\dot{x}}{2\sqrt{x}} = 0$, and hence $x = \frac{1}{3}a$, that is, AC $= \frac{1}{3}$AB.

PROBLEM 18.

Given the height of an inclined plane; required its length, so that a given power acting on a given weight, in a direction parallel to the plane, may draw it up in the least time possible.

Let a denote the height of the plane, x its length, p the power, and w the weight. Now the tendency down the plane is $= \frac{aw}{x}$, hence $p - \frac{aw}{x} =$ the motive force, and $\frac{p - \frac{aw}{x}}{p + w} = \frac{px - aw}{(p+w)x} =$ the accelerating force f; hence, by the theorems for constant forces, pa. 342 vol. 2, $t^2 = \frac{s}{gf} = \frac{(p+w)x^2}{(px - aw)g}$ must be

be a minimum, or $\frac{x^2}{px - ax}$ a min.; in fluxions, $2(px - aw)x\dot{x} - px^2\dot{x} = 0$, or $px = 2aw$, and hence $p : w :: 2a : x ::$ double the height of the plane to its length.

PROBLEM 19.

A cylinder of oak is depressed in water till its top is just level with the surface, and then is suffered to ascend; it is required to determine the greatest altitude to which it will rise, and the time of its ascent.

Let $a =$ the length, and b the area or base of the cylinder, m the specific gravity of oak, that of water being 1, also x any variable height through which the cylinder has ascended. Then, $a - x$ being the part still immersed in the water, $(a - x) \times b \times 1 = (a - x)b$ is the force of the water upwards to raise the cylinder; and $a \times b \times m = abm$ is the weight of the cylinder opposing its ascent; therefore the efficacious force to raise the cylinder is $(a - x)b - abm$; and, the mass being abm, the accelerating force is

$$\frac{(a-x)b - abm}{abm} = \frac{a - x - am}{am} = \frac{an - x}{am} = f,$$

putting $n = 1 - m$ the difference between the specific gravities of water and oak.

Now if v denote the velocity of ascent at the same time when x space is ascended, then by the theorems for variable forces, $v\dot{v} = 32f\dot{x} = \frac{32}{am} \times (an\dot{x} - x\dot{x})$, therefore

$v^2 = \frac{32}{am} \times (2anx - x^2)$, and $v = 8\sqrt{\frac{2anx - x^2}{2am}}$: but when the cylinder has acquired its greatest ascent, v and $v^2 = 0$, therefore $2anx - x^2 = 0$, and hence $x = 2an$ the part of the cylinder that rises out of the water, being $= \cdot 15a$ or $\frac{3}{20}$ of its length.

To find when the velocity is the greatest, the factor $2anx - x^2$ in the velocity must be a max. then $2an\dot{x} - 2x\dot{x} = 0$, and $x = an$, being the height above the water when the velocity is the greatest, and which it appears is just equal to the half of $2an$ above found for the greatest rise, when the upward motion ceases, and the cylinder descends again to the same depth as at first, after which it again returns ascending as before; and so on, continually playing up and down to the same highest and lowest points, like the vibrations of a pendulum, the motion ceasing in both cases in a similar manner at the extreme points, then returning, it gradually accelerates till arriving at the middle point, where it is the greatest, then gradually retarding all the way to the next

extremity of the vibration, thus making all the vibrations in equal times, to the same extent between the highest and lowest points, except that, by the small tenacity and friction &c of the water against the sides of the cylinder, it will be gradually and slowly retarded in its motion, and the extent of the vibrations decrease till at length the cylinder, like the pendulum, come to rest in the middle point of its vibrations, where it naturally floats in its quiescent state, with the part na of its length above the water.

The quantity of the greatest velocity will be found, by substituting na for x, in the general value of the velocity $8\sqrt{\frac{2anx-x^2}{2am}}$, when it becomes $8n\sqrt{\frac{a}{2m}} = \frac{4}{9}\sqrt{a}$ very nearly, the value of m being $\cdot 925$, and consequently that of $n = 1 - m = \cdot 075$.

To find the time t answering to any space x. Here $\dot{t} = \frac{\dot{x}}{v} = \frac{\dot{x}}{8\sqrt{\frac{2nax-x^2}{2ma}}} = \sqrt{\frac{ma}{32}} \times \frac{\dot{x}}{\sqrt{(2nax-x^2)}}$, and by the 13th form the fluent is $t = \frac{1}{8}\sqrt{2ma} \times A$, where A denotes the circular arc to radius 1 and versed sine $\frac{x}{na}$. Now at the middle of a vibration x is $= na$, and then the vers. $\frac{x}{na} = \frac{na}{na} = 1$ the radius, and A is the quadrantal arc $= 1\cdot 5708$; then the flu. becomes $\frac{1}{8}\sqrt{2ma} \times 1\cdot 5708 = \cdot 17\sqrt{a} \times 1\cdot 5708 = \cdot 267\sqrt{a}$ for the time of a semivibration; hence the time of each whole vibration is $\cdot 534\sqrt{a} = \frac{8}{15}\sqrt{a}$, which time therefore depends on the length of the cylinder a. To make this time $= 1$ second, a must be $= (\frac{15}{8})^2$ very nearly $= 3\frac{1}{2}$ feet or 42 inches. That is, the oaken cylinder of 42 inches length makes its vertical vibrations each in 1 second of time, or is isochronous with a common pendulum of $39\frac{1}{8}$ inches long, the extent of each vibration of the former being $6\frac{3}{10}$ inches.

PROBLEM 20.

Required to determine the quantity of matter in a sphere, the density varying as the nth power of the distance from the centre?

Let r denote the radius of the sphere, d the density at its surface, $a = 3\cdot 1416$ the area of a circle whose radius is 1, and x any distance from the centre. Then $4ax^2$ will be the surface of a sphere whose radius is x, which may be considered by expansion as generating the magnitude of the solid; therefore $4ax^2\dot{x}$ will be the fluxion of the magnitude; but

as

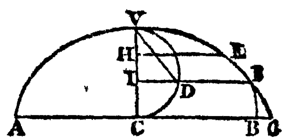
as $r^n : x^n :: d : \frac{dx^n}{r^n}$ the density at the distance x, therefore $4ax^2\dot{x} \times \frac{dx^n}{r^n} = \frac{4adx^{n+2}\dot{x}}{r^n}$ = the fluxion of the mass, the fluent of which $\frac{4adx^{n+3}}{(n+3)r^n}$, when $x = r$, is $\frac{4adr^3}{n+3}$, the quantity of the matter in the whole sphere.

Corol. 1. The magnitude of a sphere whose radius is r, being $\frac{4}{3}ar^3$, which call m; then the mass or solid content will be $\frac{3d}{n+3} \times m$, and the mean density is $\frac{3d}{n+3}$.

Corol. 2. It having been computed, from actual experiments, that the medium density of the whole mass of the earth is about $\frac{9}{5}$ times the density d at the surface, we can now determine what is the exponent of the decreasing ratio of the density from the centre to the circumference, supposing it to decrease by a regular law, viz, as r^n; for then it will be $\frac{9}{5}d = \frac{3d}{n+3}$, and hence $n = -\frac{4}{3}$. So that, in this case, the law of decrease is as $r^{-\frac{4}{3}}$, or as $\frac{1}{r^{\frac{4}{3}}}$, that is, inversely as the $\frac{4}{3}$ power of the radius.

PROBLEM 21.

Required to determine where a body, moving down the convex side of a cycloid, will fly off and quit the curve.



Let AVEB represent the cycloid, the properties of which may be seen at arts. 146 and 147 vol. 2, and VDC its generating semicircle. Let E be the point where the motion commences, whence it moves along the curve, its velocity increasing both on the curve, and also in the horizontal direction DF, till it come to such a point, F suppose, that the velocity in the latter direction is become a constant quantity, then that will be the point where it will quit the cycloid, and afterwards describe a parabola FG, because the horizontal velocity in the latter curve is always the same constant quantity, by art. 76 vol. 2.

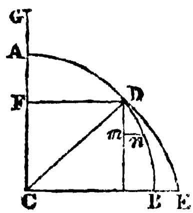
Put the diameter VC $= d$, VH $= a$, VI $= x$; then VD $= \sqrt{dx}$, and ID $= \sqrt{(dx - x^2)}$. Now the velocity in the curve at F in descending down EF, being the same as by falling through HI or $x - a$, by art. 139, will be $= 8\sqrt{(x - a)}$; but this velocity

locity in the curve at F, is to the horizontal velocity there, as VD to ID, because VD is parallel to the curve or to the tangent at F, that is $\sqrt{dx} : \sqrt{(dx - x^2)} :: 8\sqrt{(x - a)} : \frac{8\sqrt{(x-a)} \times \sqrt{(d-x)}}{\sqrt{d}}$, which is the horizontal velocity at F, where the body is supposed to have that velocity a constant quantity; therefore also $\sqrt{(x - a)} \times \sqrt{(d - x)}$, as well as $(x - a) \times (d - x) = ax + dx - ad - x^2$ is a constant quantity, and also $ax + dx - x^2$; but the fluxion of a constant quantity is equal to nothing, that is $a\dot{x} + d\dot{x} - 2x\dot{x} = 0 = a + d - 2x$, and hence $x = \frac{1}{2}a + \frac{1}{2}d =$ VI, the arithmetical mean between VH and VC.

If the motion should commence at V, then x or VI would be $= \frac{1}{2}d$, and I would be the centre of the semicircle.

PROBLEM 22.

If a body begin to move from A, *with a given velocity, along the quadrant of a circle* AB; *it is required to show at what point it will fly off from the curve.*

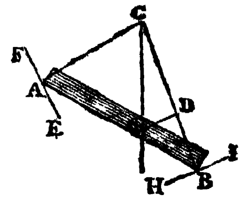


Let D denote the point where the body quits the circle ADB, and then describes the parabola DE. Draw the ordinate DF, and let GA be the height producing the velocity at A. Put GA $= a$, AC or CD $= r$, AF $= x$; then the velocity in the curve at D will be the same as that acquired by falling through GF or $a + x$, which is, as before, $8\sqrt{(a + x)}$; but the velocity in the curve is to the horizontal velocity as Dn to mn or as CD to CF by similar triangles, that is, as $r : r - x :: 8\sqrt{(x + a)} : 8\sqrt{(x + a)} \times \frac{r - x}{r}$, which is to be a constant quantity where the body leaves the circle, therefore also $(r - x)\sqrt{(x + a)}$ and $(r - x)^2 \times (x + a)$ a constant quantity; the fluxion of which made to vanish, gives $x = \frac{r - 2a}{3} =$ AF.

Hence, if $a = 0$, or the body only commence motion at A, then $x = \frac{1}{3}r$, or AF $= \frac{1}{3}$AC when it quits the circle at D. But if a or Ga were $= \frac{1}{2}r$ or $\frac{1}{2}$AC, then $r - 2a = 0$, and the body would instantly quit the circle at the vertex A, and describe a parabola circumscribing it, and having the same vertex A.

PROBLEM 23.

To determine the position of a bar or beam AB, *being supported in equilibrio by two cords* AC, BC, *having their two ends fixed in the beam, at* A *and* B.



By art. 210 vol. 2, the position will be such, that its centre of gravity G will be in the perpendicular or plumb line CG.

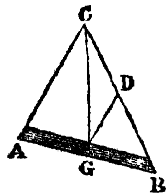
Corol. 1. Draw GD parallel to the cord AC. Then the triangle CGD, having its three sides in the directions of, or parallel to, the three forces, viz, the weight of the beam, and the tensions of the two cords AC, BC, these three forces will be proportional to the three sides CG, GD, CD, respectively, by art. 44; that is, CG is as the weight of the beam, GD as the tension or force of AC, and CD as the tension or force of BC.

Corol. 2. If two planes EAF, HBI, perpendicular to the two cords, be substituted instead of these, the beam will be still supported by the two planes, just the same as before by the cords, because the action of the planes is in the direction perpendicular to their surface; and the pressure on the planes will be just equal to the tension or force of the respective cords. So that it is the very same thing, whether the body is sustained by the two cords AC, BC, or by the two planes EF, HI; the directions and quantities of the forces acting at A and B being the same in both cases.—Also, if the body be made to vibrate about the point C, the points A, B will describe circular arcs coinciding with the touching planes at A, B; and moving the body up and down the planes, will be just the same thing as making it vibrate by the cords; consequently the body can only rest, in either case, when the centre of gravity is in the perpendicular CG.

PROBLEM 24.

To determine the position of the beam AB, *hanging by one cord* ACB, *having its ends fastened at* A *and* B, *and sliding freely over a tack or pulley fixed at* C.

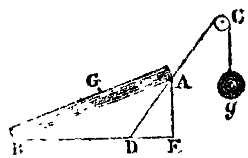
G being the centre of gravity of the beam, CG will be perpendicular to the horizon, as in the last problem. Now as the



the cord ACB moves freely about the point C, the tension of the cord is the same in every part, or the same both in AC and BC. Draw GD parallel to AC: then the sides of the triangle CGD are proportional to the three forces, the weight and the tensions of the string; that is, CD and DG are as the forces or tensions in CB and CA. But these tensions are equal; therefore CD = DG, and conseq. the opposite angles DCG and DGC are also equal: but the angle DGC is = the alternate angle ACG; theref. the angle ACG = BCG; and hence the line CG bisects the vertical angle ACB, and conseq. AC : CB :: AG : GB.

PROBLEM 25.

To determine the position of the beam AB, *moveable about the end* B, *and sustained by a given weight* g, *hanging by a cord* Ag, *going over a pulley at* C, *and fixed to the other end* A.

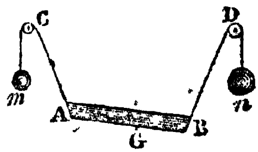


Let w = the weight of the beam, and G denote the place of its centre of gravity. Produce the direction of the cord CA to meet the horizontal line BE in D; also let fall AE perp. to BE: then AE is the direction of the weight of the beam, and DA the direction of the weight g, the former acting at G by the lever BG, and the latter at A by the lever BA; theref. the intensity of the former is $w \times$ BG, and that of the latter $g \times$ BA; but these are also proportional to the sines of their angles of direction with AB, that is, of the angles BAE, and BAD; therefore the whole intensity of the former is $w \times$ BG $\times$ sin. BAE, and of the latter it is $g \times$ BA $\times$ sin. BAD. But since these two forces balance each other, they are equal, viz, $w \times$ BG $\times$ sin. BAE $= g \times$ BA $\times$ sin. BAD, and therefore $w : g ::$ BA $\times$ sin. BAD : BG $\times$ sin. BAE, or $w \times$ BG : $g \times$ BA :: sin. BAD : sin. BAE.

PROBLEM 26.

To determine the position of the beam AB, *sustained by the given weights* m, n, *by means of the cords* ACm, BDn, *going over the fixed pulleys* C, D.

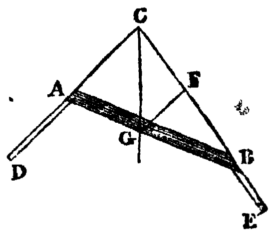
Let



Let G be the place of the centre of gravity of the beam. Now the effect of the weight m, is as m, and as the lever AG, and as the sine of the angle of direction A; and the effect of the weight n, is as n, and as the lever BG, and as the sine of the angle of direction B; but these two effects are equal, because they balance each other; that is, $m \times AG \times \sin. A = n \times BG \times \sin. B$; theref. $m \times AG : n \times BG :: \sin. B : \sin. A$.

PROBLEM 27.

To determine the position of the two posts AD *and* BE, *supporting the beam* AB, *so that the beam may rest in equilibrio.*



Through the centre of gravity G of the beam, draw CG perp. to the horizon; from any point C in which draw CAD, CBE through the extremities of the beam; then AD and BE will be the positions of the two posts or props required, so as AB may be sustained in equilibrio; because the three forces sustaining any body in such a state, must be all directed to the same point C.

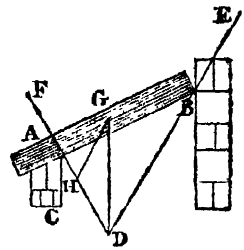
Corol. If GF be drawn parallel to CD; then the quantities of the three forces balancing the beam, will be proportional to the three sides of the triangle CGF, viz, CG as the weight of the beam, CF as the thrust or pressure in BE, and FG as the thrust or pressure in AD.

Scholium. The equilibrium may be equally maintained by the two posts or props AD, BE, as by the two cords AC, BC, or by two planes at A and B perp. to those cords.—It does not always happen that the centre of gravity is at the lowest place to which it can get, to make an equilibrium; for here when the beam AB is supported by the posts DA, EB, the centre of gravity is at the highest it can get; and being in that position, it is not disposed to move one way more than another, and therefore it is as truly in equilibrio, as if the centre was at the lowest point. It is true this is only a tottering equilibrium, and any the least force will destroy it; and then, if the beam and posts be moveable about the angles A, B, D, E, which

which is all along supposed, the beam will descend till it is below the points D, E, and gain such a position as is described in prob. 26, supposing the cords fixed at C and D, in the fig. to that prob. and then G will be at the lowest point, coming there to an equilibrium again. In planes, the centre of gravity G may be either at its highest or lowest point. And there are cases, when that centre is neither at its highest nor lowest point, as may happen in the case of prob. 24.

PROBLEM 28.

Supposing the beam AB *hanging by a pin at* B, *and lying on the wall* AC; *it is required to determine the forces or pressures at the points* A *and* B, *and their directions.*



Draw AD perp. to AB, and through G, the centre of gravity of the beam, draw GD perp. to the horizon; and join BD. Then the weight of the beam, and the two forces or pressures at A and B, will be in the directions of the three sides of the triangle ADG; or in the directions of, and proportional to, the three sides of the triangle GDH, having drawn GH parallel to BD; viz, the weight of the beam as GD, the pressure at A as HD, and the pressure at B as GH, and in these directions.

For, the action of the beam is in the direction GD; and the action of the wall at A, is in the perp. AD; conseq. the stress on the pin at B must be in the direction BD, because all the three forces sustaining a body in equilibrio, must tend to the same point, as D.

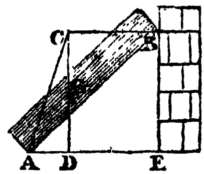
Corol. 1. If the beam were supported by a pin at A, and laid upon the wall at B; the like construction must be made at B, as has been done at A, and then the forces and their directions will be obtained.

Corol. 2. It is all the same thing, whether the beam is sustained by the pin B and the wall AC, or by two cords BE, AF, acting in the directions DB, DA, and with the forces HG, HD.

PROBLEM 29.

To determine the Quantities and Directions of the Forces, exerted by a heavy beam AB, *at its two Extremities and its Centre of Gravity, bearing against a perp. wall at its upper end* B.

From

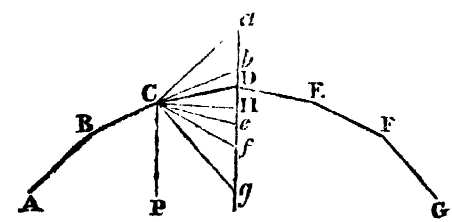


From B draw BC perp. to the face of the wall BE, which will be the direction of the force at B; also through G, the centre of gravity, draw CGD perp. to the horizontal line AE, then CD is the direction of the weight of the beam; and because these two forces meet in the point C, the third force or push A, must be in CA, directly from C; so that the three forces are in the directions CD, BC, CA, or in the directions CD, DA, CA; and, these last three forming a triangle, the three forces are not only in those directions, but are also proportional to these three lines; viz, the weight in or on the beam, as the line CD; the push against the wall at B, as the horizontal line AD; and the thrust at the bottom, as the line AC.

Some of the foregoing problems will be found useful in different cases of carpentry, especially in adapting the framing of the roofs of buildings, so as to be nearest in equilibrio in all their parts. And the last problem, in particular, will be very useful in determining the push or thrust of any arch against its piers or abutments, and thence to assign their thickness necessary to resist that push. The following problem will also be of great use in adjusting the form of a mansard roof, or of an arch, and the thickness of every part, so as to be truly balanced in a state of just equilibrium.

PROBLEM 30.

Let there be any number of lines, or bars, or beams, AB, BC, CD, DE, *&c, all in the same vertical plane, connected together and freely moveable about the joints or angles* A, B, C, D, E, *&c, and kept in equilibrio by weights laid on the angles: It is required to assign the proportion of those weights; as also the force or push in the direction of the said lines; and the horizontal thrust at every angle.*



Through any point, as D, draw a vertical line *a*DH*g* &c; to which, from any point, as C, draw lines in the direction of, or parallel to, the given lines or beams, viz, C*a* parallel to AB, and C*b* parallel to BC, and C*e* to DE, and C*f* to EF, and C*g* to FG, &c; also

also CH parallel to the horizon, or perpendicular to the vertical line *adg*, in which also all these parallels terminate.

Then will all those lines be exactly proportional to the forces acting or exerted in the directions to which they are parallel, and of all the three kinds, viz, vertical, horizontal, and oblique. That is, the oblique forces or thrusts in direction of the bars . . . . AB, BC, CD, DE, EF, FG, are proportional to their parallels *ca*, *cb*, CD, *ce*, *cf*, *cg*; and the vertical weights on the angles B, C, D, E, F, &c. are as the parts of the vertical . *ab*, *b*D, D*e*, *ef*, *fg*, and the weight of the whole frame ABCDEFG, is proportional to the sum of all the verticals, or to *ag*; also the horizontal thrust at every angle, is every where the same constant quantity, and is expressed by the constant horizontal line CH.

Demonstration. All these proportions of the forces derive and follow immediately from the general well-known property, in Statics, that when any forces balance and keep each other in equilibrio, they are respectively in proportion as the lines drawn parallel to their directions, and terminating each other.

Thus, the point or angle B is kept in equilibrio by three forces, viz, the weight laid and acting vertically downward on that point, and by the two oblique forces or thrusts of the two beams AB, CB, and in these directions. But *ca* is parallel to AB, and *cb* to BC, and *ab* to the vertical weight; these three forces are therefore proportional to the three lines *ab*, *ca*, *cb*.

In like manner, the angle C is kept in its position by the weight laid and acting vertically on it, and by the two oblique forces or thrusts in the direction of the bars BC, CD: consequently these three forces are proportional to the three lines *b*D, *cb*, CD, which are parallel to them.

Also, the three forces keeping the point D in its position, are proportional to their three parallel lines, D*e*, CD, *ce*. And the three forces balancing the angle E, are proportional to their three parallel lines *ef*, *ce*, *cf*. And the three forces balancing the angle F, are proportional to their three parallel lines *fg*, *cf*, *cg*. And so on continually, the oblique forces or thrust in the directions of the bars or beams, being always proportional to the parts of the lines parallel to them, intercepted by the common vertical line; while the vertical forces or weights, acting or laid on the angles, are proportional to the parts of this vertical line intercepted by the two lines parallel to the lines of the corresponding angles.

Again, with regard to the horizontal force or thrust: since the

the line DC represents, or is proportional to the force in the direction DC, arising from the weight or pressure on the angle D; and since the oblique force DC is equivalent to, and resolves into, the two DH, HC, and in those directions, by the resolution of forces, viz, the vertical force DH, and the horizontal force HC; it follows, that the horizontal force or thrust at the angle D, is proportional to the line CH; and the part of the vertical force or weight on the angle D, which produces the oblique force DC, is proportional to the part of the vertical line DH.

In like manner, the oblique force c*b*, acting at C, in the direction CB, resolves into the two *b*H, HC; therefore the horizontal force or thrust at the angle C, is expressed by the line CH, the very same as it was before for the angle D; and the vertical pressure at C, arising from the weights on both D and C, is denoted by the vertical line *b*H.

Also, the oblique force *a*C, acting at the angle B, in the direction BA, resolves into the two *a*H, HC; therefore again the horizontal thrust at the angle B, is represented by the line CH, the very same as it was at the points C and D; and the vertical pressure at B, arising from the weights on B, C, and D, is expressed by the part of the vertical line *a*H.

Thus also, the oblique force C*e*, in direction DE, resolves into the two CH, H*e*, being the same horizontal force, with the vertical H*e*; and the oblique force C*f*, in direction EF, resolves into the two CH, H*f*; and the oblique force C*g*, in direction FG, resolves into the two CH, H*g*; and the oblique force C*g*, in direction FG, resolves into the two CH, H*g*; and so on continually, the horizontal force at every point being expressed by the same constant line CH; and the vertical pressures on the angles by the parts of the verticals, viz, *a*H the whole vertical pressure at B, from the weights on the angle B, C, D: and *b*H the whole pressure on C from the weights on C and D; and DH the part of the weight on D causing the oblique force DC; and H*e* the other part of the weight on D causing the oblique pressure DE; and H*f* the whole vertical pressure at E from the weights on D and E; and H*g* the whole vertical pressure on F arising from the weights laid on D, E, and F. And so on.

So that, on the whole, *a*H denotes the whole weight on the points from D to A; and H*g* the whole weight on the points from D to G; and *ag* the whole weight on all the points on both sides; while *ab*, *b*D, D*e*, *ef*, *fg* express the several particular weights, laid on the angles B, C, D, E, F.

Also, the horizontal thrust is everywhere the same constant quantity, and is denoted by the line CH.

*. Lastly,

Lastly, the several oblique forces or thrusts, in the directions AB, BC, CD, DE, EF, FG, are expressed by, or are proportional to, their corresponding parallel lines, c*a*, c*b*, CD, c*e*, c*f*, c*g*.

Corol. 1. It is obvious, and remarkable, that the lengths of the bars AB, BC, &c, do not affect or alter the proportions of any of these loads or thrusts; since all the lines c*a*, c*b*, *ab*, &c, remain the same, whatever be the lengths of AB, BC, &c. The positions of the bars, and the weights on the angles depending mutually on each other, as well as the horizontal and oblique thrusts. Thus, if there be given the position of DC, and the weights or loads laid on the angles D, C, B; set these on the vertical, DH, D*b*, *ba*, then c*b*, c*a* give the directions or positions of CB, BA, as well as the quantity or proportion CH of the constant horizontal thrust.

Corol. 2. If CH be made radius; then it is evident that H*a* is the tangent, and c*a* the secant of the elevation of c*a* or AB above the horizon; also H*b* is the tangent and c*b* the secant of the elevation of c*b* or CB; also HD and CD the tangent and secant of the elevation of CD; also H*e* and c*e* the tangent and secant of the elevation of c*e* or DE; also H*f* and c*f* the tangent and secant of the elevation of EF; and so on; also the parts of the vertical *ab*, *b*D, *ef*, *fg*, denoting the weights laid on the several angles, are the differences of the said tangents of elevations. Hence then in general,

1st. The oblique thrusts, in the directions of the bars, are to one another, directly in proportion as the secants of their angles of elevation above the horizontal directions; or, which is the same thing, reciprocally proportional to the cosines of the same elevations, or reciprocally proportional to the sines of the vertical angles, *a*, *b*, D, *e*, *f*, *g*, &c, made by the vertical line with the several directions of the bars; because the secants of any angles are always reciprocally in proportion as their cosines.

2. The weight or load laid on each angle, is directly proportional to the difference between the tangents of the elevations above the horizon, of the two lines which form the angle.

3. The horizontal thrust at every angle, is the same constant quantity, and has the same proportion to the weight on the top of the uppermost bar, as radius has to the tangent of the elevation of that bar. Or, as the whole vertical *ag*, is to the line CH, so is the weight of the whole assemblage of bars, to the horizontal thrust. Other properties also, concerning the weights and the thrusts, might be pointed out, but they are less simple and elegant than the above, and are therefore omitted;

omitted; the following only excepted, which are inserted here on account of their usefulness.

Corol. 3. It may hence be deduced also, that the weight or pressure laid on any angle, is directly proportional to the continual product of the sine of that angle and of the secants of the elevations of the bars or lines which form it. Thus, in the triangle bCD, in which the side bD is proportional to the weight laid on the angle C, because the sides of any triangle are to one another as the sines of their opposite angles, therefore as sin. D : Cb : : sin. bCD : bD; that is, bD is as $\frac{\text{sin. } b\text{CD}}{\text{sin. D}} \times \text{C}b$; but the sine of angle D is the cosine of the elevation DCH, and the cosine of any angle is reciprocally proportional to the secant, therefore bD is as sin. bCD $\times$ sec. DCH $\times$ Cb; and Cb being as the secant of the angle bCH of the elevation of bC or BC above the horizon, therefore bD is as sin. bCD $\times$ sec. bCH $\times$ sec. DCH; and the sine of bCD being the same as the sine of its supplement BCD; therefore the weight on the angle C, which is as bD, is as the sin. BCD $\times$ sec. DCH $\times$ sec. bCH, that is, as the continual product of the sine of that angle, and the secants of the elevations of its two sides above the horizon.

Corol. 4. Further, it easily appears also, that the same weight on any angle C, is directly proportional to the sine of that angle BCD, and inversely proportional to the sines of the two parts BCP, DCP, into which the same angle is divided by the vertical line CP. For the secants of angles are reciprocally proportional to their cosines or sines of their complements: but BCP $=$ CbH, is the complement of the elevation bCH, and DCP is the complement of the elevation DCH; therefore the secant of bCH $\times$ secant of DCH is reciprocally as the sin. BCP $\times$ sin. DCP; also the sine of bCD is $=$ the sine of its supplement BCD; consequently the weight on the angle C, which is proportional to sin. bCD $\times$ sec. bCH $\times$ sec. DCH, is also proportional to $\frac{\text{sin. BCD}}{\text{sin. BCP} \times \text{sin. DCP}}$, when the whole frame or series of angles is balanced, or kept in equilibrio, by the weights on the angles; the same as in the preceding proposition.

Scholium. The foregoing proposition is very fruitful in its practical consequences, and contains the whole theory of arches, which may be deduced from the premises by supposing the constituting bars to become very short, like arch stones, so as to form the curve of an arch. It appears too, that the horizontal thrust, which is constant or uniformly the same

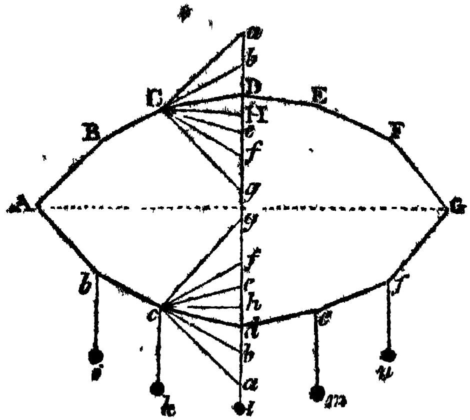
same throughout, is a proper measuring unit, by means of which to estimate the other thrusts and pressures, as they are all determinable from it and the given positions; and the value of it, as appears above, may be easily computed from the uppermost or vertical part alone, or from the whole assemblage together, or from any part of the whole, counted from the top downwards.

The solution of the foregoing proposition depends on this consideration, viz, that an assemblage of bars or beams, being connected together by joints at their extremities, and freely movable about them, may be placed in such a vertical position, as to be exactly balanced, or kept in equilibrio, by their mutual thrusts and pressures at the joints; and that the effect will be the same if the bars themselves be considered as without weight, and the angles be pressed down by laying on them weights which shall be equal to the vertical pressures at the same angles, produced by the bars in the case when they are considered as endued with their own natural weights. And as we have found that the bars may be of any length, without affecting the general properties and proportions of the thrusts and pressures, therefore by supposing them to become short, like arch stones, it is plain that we shall then have the same principles and properties accommodated to a real arch of equilibration, or one that supports itself in a perfect balance. It may be further observed, that the conclusions here derived, in this proposition and its corollaries, exactly agree with those derived in a very different way, in my principles of bridges, viz, in propositions 1 and 2, and their corollaries.

PROBLEM 31.

If the whole figure in the last problem be inverted, or turned round the horizontal line AG *as an axis, till it be completely reversed, or in the same vertical plane below the first position, each angle* D, *d, &c, being in the same plumb line; and if weights i, k, l, m, n, which are respectively equal to the weights laid on the angles* B, C, D, E, F, *of the first figure, be now suspended by threads from the corresponding angles b, c, d, e, f, of the lower figure; it is required to show that those weights keep this figure in exact equilibrio, the same as the former, and all the tensions or forces in the latter case, whether vertical or horizontal or oblique, will be exactly equal to the corresponding forces of weight or pressure or thrust in the like directions of the first figure.*

This

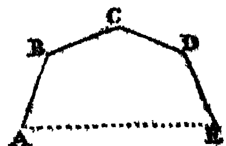


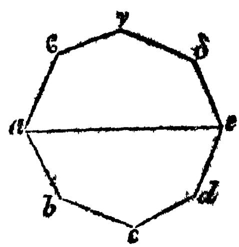
This necessarily happens, from the equality of the weights, and the similarity of the positions, and actions of the whole in both cases. Thus, from the equality of the corresponding weights, at the like angles, the ratios of the weights, *ab*, *bd*, *dh*, *he*, &c, in the lower figure, are the very same as those, *ab*, *b*D, DH, H*e*, &c, in the upper figure; and from the equality of the constant horizontal forces CH, *ch*, and the similarity of the positions, the corresponding vertical lines, denoting the weights, are equal, namely, $ab = ab$, $b\text{D} = bd$, $\text{DH} = dh$, &c. The same may be said of the oblique lines also, *ca*, *cb*, &c, which being parallel to the beams A*b*, *bc*, &c, will denote the tensions of these, in the direction of their length, the same as the oblique thrusts or pushes in the upper figures. Thus, all the corresponding weights and actions, and positions, in the two situations, being exactly equal and similar, changing only drawing and tension for pushing and thrusting, the balance and equilibrium of the upper figure is still preserved the same in the hanging festoon or lower one.

Scholium. The same figure, it is evident, will also arise, if the same weights, *i*, *k*, *l*, *m*, *n*, be suspended at like distances A*b*, *bc*, &c, on a thread, or cord, or chain, &c, having in itself little or no weight. For the equality of the weights, and their directions and distances, will put the whole line, when they come to equilibrium, into the same festoon shape of figure. So that, whatever properties are inferred in the corollaries to the foregoing prob. will equally apply to the festoon or lower figure hanging in equilibrio.

This is a most useful principle in all cases of equilibriums, especially to the mere practical mechanist, and enables him in an experimental way to resolve problems, which the best mathematicians have found it no easy matter to effect by

mere computation. For thus, in a simple and easy way he obtains the shape of an equilibrated arch or bridge; and thus also he readily obtains the positions of the rafters in the frame of an equilibrated curb or mansard roof; a single instance of which may serve to show the extent and uses to which it may be applied Thus, if it should be required to make a curb frame roof having a given width AE, and consisting of four rafters AB, BC, CD, DE, which shall either be equal or in any given proportion to each other. There can be no doubt but that the best form of the roof will be that which puts all its parts in equilibrio, so that there may be no unbalanced parts which may require the aid of ties or stays to keep the frame in its position Here the mechanic has nothing to do, but to take four like but small pieces, that are either equal or in the same given proportions as those proposed, and connect them closely together at the joints A, B, C, D, E, by pins or strings, so as to be freely moveable about them; then suspend this from two pins *a*, *e*, fixed in a horizontal line, and the chain of the pieces will arrange itself in such a festoon or form, *abcde*, that all its parts will come to rest in equilibrio. Then, by inverting the figure, it will exhibit the form and frame of a curb roof *αβγδε*, which will also be in equilibrio, the thrusts of the pieces now balancing each other, in the same manner as was done by the mutual pulls or tensions of the hanging festoon *a b c d e*. By varying the distance *ae*, of the points of suspension, moving them nearer to, or farther off, the chain will take different forms; then the frame ABCDE may be made similar to that form which has the most pleasing or convenient shape, found above as a model.





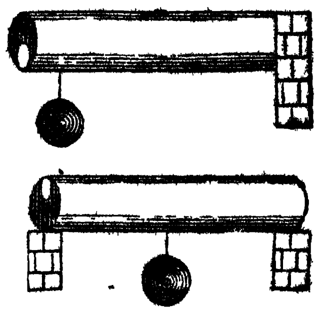
Indeed this principle is exceeding fruitful in its practical consequences. It is easy to perceive that it contains the whole theory of the construction of arches: for each stone of an arch may be considered as one of the rafters or beams in the foregoing frames, since the whole is sustained by the mere principle of equilibration, and the method, in its application, will afford some elegant and simple solutions of the most difficult cases of this important problem.

PROBLEM

PROBLEM 32.

Of all Hollow Cylinders, whose Lengths and the Diameters of the Inner and Outer Circles continue the same, it is required to show what will be the Position of the Inner Circle when the Cylinder is the Strongest Laterally.

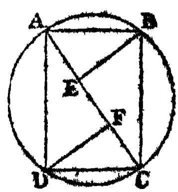
Since the magnitude of the two circles are constant, the area of the solid space, included between their two circumferences, will be the same, whatever be the position of the inner circle, that is, there is the same number of fibres to be broken, and in this respect the strength will be always the same. The strength then can only vary according to the situation of the centre of gravity of the solid part, and this again will depend on the place where the cylinder must first break, or on the manner in which it is fixed.



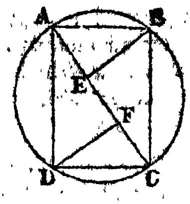
Now, by cor. 8 art. 251 v. 2, the cylinder is strongest when the hollow, or inner circle, is nearest to that side where the fracture is to end, that is, at the bottom when it breaks first at the upper side, or when the cylinder is fixed only at one end as in the first figure. But the reverse will be the case when the cylinder is fixed at both ends; and consequently when it opens first below, or ends above, as in the 2d figure annexed.

PROBLEM 33.

To determine the Dimensions of the Strongest Rectangular Beam that can be cut out of a Given Cylinder.



Let AB, the breadth of the required beam, be denoted by b, AD the depth by d, and the diameter AC of the cylinder by D. Now when AB is horizontal, the lateral strength is denoted by bd^2 (by art. 248 vol. 2), which is to be a maximum. But $AD^2 = AC^2 - AB^2$, or $d^2 = D^2 - b^2$; theref. $bd^2 = (D^2 - b^2)b = D^2b - b^3$ is a maximum; in fluxions $D^2\dot{b} - 3b^2\dot{b} = 0 = D^2 - 3b^2$, or $D^2 = 3b^2$; also $d^2 = D^2 - b^2 = 3b^2 - b^2 = 2b^2$. Conseq. $b^2 : d^2 : D^2 :: 1 : 2 : 3$, that is, the squares of the breadth, and of the depth, and of the cylinder's diameter, are to one another respectively as the three numbers 1, 2, 3.



Corol. 1. Hence results this easy practical construction: divide the diameter AC into three equal parts, at the points E, F; erect the perpendiculars EB, FD; and join the points B, D to the extremities of the diameter: so shall ABCD be the rectangular end of the beam as required. For, because AE, AB, AC are in continued proportion (theor. 87 Geom.), theref. AE : AC :: AB^2 : AC^2; and in like manner AF : AC :: AD^2 : AC^2; hence AE : AF : AC :: AB^2 : AD^2 : AC^2 :: 1 : 2 : 3.

Corol. 2. The ratios of the three b, d, D, being as the three $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, or as 1, 1·414, 1·732, are nearly as the three 5, 7, 8·6, or more nearly as 12, 17, 20·8.

Corol. 3. A square beam cut out of the same cylinder, would have its side $= D\sqrt{\frac{1}{2}} = \frac{1}{2}D\sqrt{2}$. And its solidity would be to that of the strongest beam, as $\frac{1}{2}D^2$ to $\frac{1}{3}D^2\sqrt{2}$, or as 3 to $2\sqrt{2}$, or as 3 to 2·828; while its strength would be to that of the strongest beam, as $(D\sqrt{\frac{1}{2}})^3$ to $D\sqrt{\frac{1}{3}} \times \frac{2}{3}D^2$, or as $\frac{1}{4}\sqrt{2}$ to $\frac{2}{9}\sqrt{3}$, or as $9\sqrt{2}$ to $8\sqrt{3}$, or nearly as 101 to 110.

Corol. 4. Either of these beams will exert the greatest lateral strength, when the diagonal of its end is placed vertically, by art. 252 vol. 2.

Corol. 5. The strength of the whole cylinder will be to that of the square beam, when placed with its diagonal vertically, as the area of the circle to that of its inscribed square. For, the centre of the circle will be the centre of gravity of both beams, and is at the distance of the radius from the lowest point in each of them; conseq their strengths will be as their areas, by art. 243 vol. 2.

PROBLEM 34.

To determine the Difference in the Strength of a Triangular Beam, according as it lies with the Edge or with the Flat Side Upwards.

In the same beam, the area is the same, and therefore the strength can only vary with the distance of the centre of gravity from the highest or lowest point; but, in a triangle, the distance of the centre of gravity from an angle, is double of its distance from the opposite side; therefore the strength of the beam will be as 2 to 1 with the different sides upwards, under different circumstances, viz, when the centre of gravity is farthest from the place where fracture ends, by art. 243 vol. 2; that is, with the angle upwards when the beam is

supported

supported at both ends; but with the side upwards, when it is supported only at one end, (art. 252 vol. 2) because in the former case the beam breaks first below, but the reverse in the latter case.

PROBLEM 35.

Given the Length and Weight of a Cylinder or Prism, placed Horizontally with one end firmly fixed, and will just support a given weight at the other end without breaking; it is required to find the Length of a similar Prism or Cylinder which, when supported in like manner at one end, shall just bear without breaking another given weight at the unsupported end.

Let l denote the length of the given cylinder or prism, d the diameter or depth of its end, w its weight, and u the weight hanging at the unsupported end; also let the like capitals L, D, W, U, denote the corresponding particulars of the other prism or cylinder. Then, the weights of similar solids of the same matter being as the cubes of their lengths, as $l^3 : L^3 :: w : \frac{L^3}{l^3}w$, the weight of the prism whose length is L. Now $\frac{1}{2}wl$ will be the stress on the first beam by its own weight w acting at its centre of gravity, or at half its length; and lu the stress of the added weight u at its extremity, their sum $(\frac{1}{2}w + u)l$ will therefore be the whole stress on the given beam: in like manner the whole stress on the other beam, whose weight is W or $\frac{L^3}{l^3}w$, will be $(\frac{1}{2}W + U)L$ or $(\frac{L^3}{2l^3}w + U)L$.

But the lateral strength of the first beam is to that of the second, as d^3 to D^3 (art. 246 vol. 2), or as l^3 to L^3; and the strengths and stresses of the two beams must be in the same ratio, to answer the conditions of the problem; therefore as $(\frac{1}{2}w + u)l : (\frac{L^3}{2l^3}w + U)L :: l^3 : L^3$; this analogy, turned into an equation, gives $L^3 - \frac{w+2u}{w}lL^2 + \frac{2}{w}l^3U = 0$, a cubic equation, from which the numeral value of L may be easily determined, when those of the other letters are known.

Corol. 1. When U vanishes, the equation gives $L^3 = \frac{w+2u}{w}lL^2$, or $L = \frac{w+2u}{w}l$, whence $w : w + 2u :: l : L$, for the length of the beam, which will but just support its own weight.

Corol. 2. If a beam just only support its own weight, when fixed at one end; then a beam of double its length, fixed at both ends, will also just sustain itself: or if the one just break, the other will do the same.

PROBLEM

PROBLEM 36.

Given the Length and Weight of a Cylinder or Prism, fixed Horizontally as in the foregoing problem, and a weight which, when hung at a given point, Breaks the Prism: it is required to determine how much longer the Prism, of equal Diameter or of equal Breadth and Depth, may be extended before it Break, either by its own weight, or by the addition of any other adventitious weight.

Let l denote the length of the given prism, w its weight, and u a weight attached to it at the distance d from the fixed end; also let L denote the required length of the other prism, and U the weight attached to it at the distance D. Now the strain occasioned by the weight of the first beam is $\frac{1}{2}wl$, and that by the weight u at the distance d, is du, their sum $\frac{1}{2}wl + du$ being the whole strain. In like manner $\frac{1}{2}WL + DU$ is the strain on the second beam; but $l : L :: w : \frac{Lw}{l} = W$ the weight of this beam, theref. $\frac{wL^2}{2l} + DU =$ its strain. But the strength of the beam, which is just sufficient to resist these strains, is the same in both cases; therefore $\frac{wL^2}{2l} + DU = \frac{1}{2}wl + du$, and hence, by reduction, the required length $L = \sqrt{(l \times \frac{wl + 2du - 2DU}{w})}$.

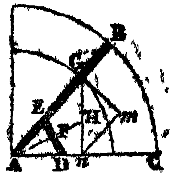
Corol. 1. When the lengthened beam just breaks by its own weight, then $U = 0$ or vanishes, and the required length becomes $L = \sqrt{(l \times \frac{wl + 2du}{w})}$.

Corol. 2. Also when U vanishes, if d become $= l$, then $L = l\sqrt{\frac{w + 2u}{w}}$ is the required length.

PROBLEM 37.

Let AB *be a beam moveable about the end* A, *so as to make any angle* BAC *with the plane of the horizon* AC: *it is required to determine the position of a prop or supporter* DE *of a given length, which shall sustain it with the greatest ease in any given position; also to ascertain the angle* BAC *when the least force which can sustain* AB, *is greater than the least force in any other position.*

Let



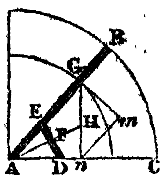
Let G be the centre of gravity of the beam; and draw Gm perp. to AB, Gn to AC, nm to Gm, and AFH to DE. Put $r =$ AG, $p =$ DE, $w =$ the weight of the beam AB, and A$n = x$. Then, by the nature of the parallelogram of forces, Gn : Gm, or by sim. triangles, AG $= r$: A$n = x$ :: w : $\frac{wx}{r}$, the force which acting at G in the direction mG, is sufficient to sustain the beam; and, by the nature of the lever, AE : AG $= r$ :: $\frac{xw}{AG}$ the requisite force at G : $\frac{wx}{AE}$, the force capable of supporting it at E in a direction perp. to AB or parallel to mG; and again as AF : AE :: $\frac{wx}{AE} . \frac{wx}{AF}$, the force or pressure actually sustained by the given prop DE in a direction perp. to AF. And this latter force will manifestly be the least possible when the perp. AF upon DE is the greatest possible, whatever the angle BAC may be, which is when the triangle ADE is isosceles, or has the side AD = AE, by an obvious corol. from the latter part of prob. 6 pa. 171 of this volume.

Secondly, for a solution to the latter part of the problem, we have to find when $\frac{ux}{AF}$ is a maximum; the angles D and F being always equal to each other, while they vary in magnitude by the change in the position of AB. Let AF produced meet Gn in H: then, in the similar triangles ADF, AHn, it will be AF : A$n = x$ :: DF $= \frac{1}{2}p$: Hn, hence $\frac{x}{AF} = \frac{Hn}{\frac{1}{2}p}$, and conseq. $\frac{x}{F} \times w = \frac{Hn}{\frac{1}{2}p} \times w$. But, by theor. 83 Geom. and comp. AG + A$n = r + x$: A$n = x$ :: G$n = \sqrt{(r^2 - x^2)}$: H$n = \frac{x}{r+x}\sqrt{(r^2 - x^2)} = x\sqrt{\frac{r-x}{r+x}}$: consequently the force $\frac{Hn}{\frac{1}{2}p} \times w$, acting on the prop, is also truly expressed by $\frac{ux}{\frac{1}{2}p}\sqrt{\frac{r-x}{r+x}}$. Then the fluxion of this made to vanish gives $x = \frac{\sqrt{5}-1}{2}r$ the cos. angle BAC $= 51° 50'$, the inclination required.

PROBLEM

PROBLEM 38.

Suppose the Beam AB, *instead of being moveable about the centre* A, *as in the last problem, to be supported in a given position by means of the given prop* DE: *it is required to determine the position of that prop, so that the prismatic beam* AC, *on which it stands, may be the least liable to breaking, this latter beam being only supported at its two ends* A *and* C.



Put the base AC $= b$, the prop. DE $= p$, AG $= r$, the weight of AB $= w$, s and c the sine and cosine of $\angle$ A, $x =$ sin. $\angle$ E, $y =$ sin. $\angle$ D, and $z =$ AE. Then, by trigon. $z : y :: p : s$, or $\frac{y}{z} = \frac{s}{p}$, and AD $= \frac{px}{s}$; also $cw =$ the force of the beam at G in direction Gm. Let F denote the force sustaining the beam at E in the direction ED: then, because action and reaction are equal and opposite, the same force will be exerted at D in the direction DE: therefore AG . $cw =$ Fzx, and F $= \frac{rcw}{zx}$. Again, the vertical stress at D, will be as F $\times$ sine D $\times$ AD . DC $=$ Fy . AD . DC $= \frac{rcwy}{zx} \times \frac{px}{s}(b - \frac{px}{s}) =$ (substituting $\frac{s}{p}$ for its equal $\frac{y}{z}$) $\frac{rcws}{px} \times \frac{px}{s} \times \frac{bs-px}{s} = rcw \times \frac{bs-px}{s} = \frac{rcwp}{s} \times (\frac{bs}{p} - x) =$ a minimum by the problem. Conseq. $\frac{bs}{p} - x$ is a minimum, or x a maximum, that is, $x = 1$, and the angle E is a right angle. Hence the point E is easily found by this proportion, sin. A : cos. A :: ED : EA.

PROBLEM 39.

To explain the Disposition of the Parts of Machines.

When several pieces of timber, iron, or any other materials, are employed in a machine or structure of any kind, all the parts, both of the same piece, and of the different pieces in the fabric, ought to be so adjusted with respect to magnitude, that the strength in every part may be, as near as possible, in a constant proportion to the stress or strain to which they will be subjected. Thus, in the construction of any engine, the weight and pressure on every part should be investigated, and the strength apportioned accordingly. All levers, for instance, should be made strongest where they are most strained: viz, levers of the first kind, at the fulcrum; levers of

of the second kind, where the weight acts; and those of the third kind, where the power is applied. The axles of wheels and pulleys, the teeth of wheels, also ropes, &c, must be made stronger or weaker, as they are to be more or less acted on. The strength allotted should be more than fully competent to the stress to which the parts can ever be liable; but without allowing the surplus to be extravagant: for an over excess of strength in any part, instead of being serviceable, would be very injurious, by increasing the resistance the machine has to overcome, and thus encumbering, impeding, and even preventing the requisite motion; while, on the other hand, a defect of strength in any part will cause a failure there, and either render the whole useless, or demand very frequent repairs.

PROBLEM 40.

To ascertain the Strength of Various Substances.

The proportions that we have given on the strength and stress of materials, however true, according to the principles assumed, are of little or no use in practice, till the comparative strength of different substances is ascertained: and even then they will apply more or less accurately to different substances. Hitherto they have been applied almost exclusively to the resisting force of beams of timber; though probably no materials whatever accord less with the theory than timber of all kinds. In the theory, the resisting body is supposed to be perfectly homogeneous, or composed of parallel fibres, equally distributed round an axis, and presenting uniform resistance to rupture. But this is not the case in a beam of timber: for, by tracing the process of vegetation, it is readily seen that the ligneous coats of a tree, formed by its annual growth, are almost concentric; being like so many hollow cylinders thrust into each other, and united by a kind of medullary substance, which offers but little resistance: these hollow cylinders therefore furnish the chief strength and resistance to the force which tends to break them.

Now, when the trunk of a tree is squared, in order that it may be converted into a beam, it is plain that all the ligneous cylinders greater than the circle inscribed in the square or rectangle, which is the transverse section of the beam, are cut off at the sides; and therefore almost the whole strength or resistance arises from the cylindric trunk inscribed in the solid part of the beam; the portions of the cylindric coats, situated towards the angles, adding but little comparatively to the strength and resistance of the beam. Hence it follows that we cannot, by legitimate comparison, accurately deduce the

the strength of a joist, cut from a small tree, by experiments on another which has been sawn from a much larger tree or block. As to the concentric cylinders above mentioned, they are evidently not all of equal strength: those nearest the centre, being the oldest, are also the hardest and strongest; which again is contrary to the theory, in which they are supposed uniform throughout. But yet, after all however, it is still found that, in some of the most important problems, the results of the theory and well-conducted experiments coincide, even with regard to timber: thus, for example, the experiments on rectangular beams afford results deviating but in a very slight degree from the theorem, that the strength is proportional to the product of the breadth and the square of the depth.

Experiments on the strength of different kinds of wood, are by no means so numerous as might be wished: the most useful seem to be those made by Muschenbroek, Buffon, Emerson, Parent, Banks, and Girard. But it will be at all times highly advantageous to make new experiments on the same subject; a labour especially reserved for engineers who possess skill and zeal for the advancement of their profession. It has been found by experiments, that the same kind of wood, and of the same shape and dimensions, will bear or break with very different weights: that one piece is much stronger than another, not only cut out of the same tree, but out of the same rod; and that even, if a piece of any length, planed equally thick throughout, be separated into three or four pieces of an equal length, it will often be found that these pieces require different weights to break them. Emerson observes that wood from the boughs and branches of trees is far weaker than that of the trunk or body; the wood of the large limbs stronger than that of the smaller ones; and the wood in the heart of a sound tree strongest of all; though some authors differ on this point. It is also observed that a piece of timber which has borne a great weight for a short time, has broke with a far less weight, when left upon it for a much longer time. Wood is also weaker when green, and strongest when thoroughly dried, in the course of two or three years, at least. Wood is often very much weakened by knots in it; also when cross-grained, as often happens in sawing, it will be weakened in a greater or less degree, according as the cut runs more or less across the grain. From all which it follows, that a considerable allowance ought to be made for the various strength of wood, when applied to any use where strength and durability are required.

Iron is much more uniform in its strength than wood. Yet

experiments

experiments show that there is some difference arising from different kinds of ore: a difference is also found not only in iron from different furnaces, but from the same furnace, and even from the same melting; which may arise in a great measure from the different degrees of heat it has when poured into the mould.

Every beam or bar, whether of wood, iron, or stone, is more easily broken by any transverse strain, while it is also suffering any very great compression endways; so much so indeed that we have sometimes seen a rod, or a long slender beam, when used as a prop or shoar, urged home to such a degree, that it has burst asunder with a violent spring. Several experiments have been made on this kind of strain: a piece of white marble, $\frac{1}{4}$ of an inch square, and 3 inches long, bore 38 lbs; but when compressed endways with 300 lbs, it broke with $14\frac{1}{2}$lbs. The effect is much more observable in timber, and more elastic bodies; but is considerable in all. This is a point therefore that must be attended to in all experiments; as well as the following, viz, that a beam supported at both ends, will carry almost twice as much when the ends beyond the props are kept from rising, as when the beam rests loosely on the props.

The following list of the absolute strength of several materials, is extracted from the collection made by professor Robison, from the experiments of Muschenbroek and other experimentalists. The specimens are supposed to be prisms or cylinders of one square inch transverse area, which are stretched or drawn lengthways by suspended weights, gradually increased till the bars parted or were torn asunder, by the number of avoirdupois pounds, on a medium of many trials, set opposite each name.

1st. Metals.

	lbs.		lbs.
Gold, cast	22,000	Tin, cast	5,000
Silver, cast	42,000	Lead, cast	860
Copper, cast . . .	34,000	Regulus of Antimony	1,000
Iron, cast	50,000	Zinc	2,600
Iron, bar	70,000	Bismuth	2,900
Steel, bar	135,000		

It is very remarkable that almost all the metallic mixtures are more tenacious than the metals themselves. The change of tenacity depends much on the proportion of the ingredients; and yet the proportion which produces the most tenacious mixture, is different in the different metals. The proportion of

of ingredients here selected, is that which produces the greatest strength.

	lbs.		lbs.
2 parts gold with 1 silver	28,000	Brass, of copper & tin	51,000
		3 tin, 1 lead	10,200
5 pts gold, 1 copper	50,000	8 tin, 1 zinc	10,000
5 silver, 1 copper	48,500	4 tin, 1 regul. antim.	12,000
4 silver, 1 tin	41,000	8 lead, 1 zinc	4,500
6 copper, 1 tin	60,000	4 tin, 1 lead, 1 zinc	13,000

These numbers are of considerable use in the arts. The mixtures of copper and tin are particularly interesting in the fabric of great guns. By mixing copper, whose greatest strength does not exceed 37,000, with tin which does not exceed 6000, is produced a metal whose tenacity is almost double, at the same time that it is harder and more easily wrought: it is however more fusible. We see also that a very small addition of zinc almost doubles the tenacity of tin, and increases the tenacity of lead 5 times; and a small addition of lead doubles the tenacity of tin. These are economical mixtures; and afford valuable information to plumbers for augmenting the strength of water-pipes. Also, by having recourse to these tables, the engineer can proportion the thickness of his pipes, of whatever metal, to the pressures they are to suffer.

2d. Woods, &c.

	lbs.		lbs.
Locust tree	20,100	Tamarind	8,750
Jujeb	18,500	Fir	8,330
Beech, Oak	17,300	Walnut	8,130
Orange	15,500	Pitch pine	7,650
Alder	13,900	Quince	6,750
Elm	13,200	Cypress	6,000
Mulberry	12,500	Poplar	5,500
Willow	12,500	Cedar	4,880
Ash	12,000	Ivory	16,270
Plum	11,800	Bone	5,250
Elder	10,000	Horn	8,750
Pomegranate	9,750	Whalebone	7,500
Lemon	9,250	Tooth of sea-calf	4,075

It is to be observed that these numbers express something more than the utmost cohesion; the weights being such as will very soon, perhaps, in a minute or two, tear the rods asunder. It may be said in general, that $\frac{2}{3}$ of these weights will sensibly impair the strength, after acting a considerable while, and that one-half is the utmost that can remain permanently

manently suspended at the rods with safety; and it is this last allotment that the engineer should reckon upon in his constructions. There is however considerable difference in this respect: woods of a very straight fibre, such as fir, will be less impaired by any load which is not sufficient to break them immediately. According to Mr. Emerson, the load which may be safely suspended to an inch square of various materials, is as follows.

	lbs.		lbs.
Iron	76,400	Red fir, holly, elder, plane	5,000
Brass	35,600	Cherry, hazle . .	4,760
Hempen rope . . .	19,600	Alder, asp, birch, willow	4,290
Ivory	15,700	Freestone	914
Oak, box, yew, plumb	7,850	Lead	430
Elm, ash, beech .	6,070		
Walnut, plum . .	5,360		

He gives also the practical rule, that a cylinder whose diameter is d inches, loaded to $\frac{1}{4}$ of its absolute strength, will carry permanently as here annexed.

	cwts.
Iron	$135d^2$
Good rope .	$22d^2$
Oak	$14d^2$
Fir	$9d^2$

Experiments on the transverse strength of bodies are easily made, and accordingly are very numerous, especially those made on timber, being the most common and the most interesting. The completest series we have seen is that given by Belidor, in his Science des Ingenieurs, and is exhibited in the following table. The first column simply indicates the number of the experiments; the column b shows the breadth of the pieces, in inches; the column d contains their depths; the column l shows the lengths; and column *lbs* shows the weights in pounds which broke them, when suspended by their middle points, being the medium of 3 trials of each piece; the accompanying words, *fixed* and *loose* denoting whether the ends were firmly fixed down, or simply lay loose on the supports.

N°.	b	d	l	*lbs*	
1	1	1	18	406	loose.
2	1	1	18	608	fixed.
3	2	1	18	805	loose.
4	1	2	18	1580	loose.
5	1	1	36	187	loose.
6	1	1	36	283	fixed.
7	2	2	36	1585	loose.
8	$1\frac{2}{3}$	$2\frac{1}{3}$	36	1660	loose.

By

By comparing experiments 1 and 3, the strength appears proportional to the breadth.

Experiments 3 and 4 show the strength to be as the breadth multiplied by the square of the depth.

Experiments 1 and 5 show the strength nearly in the inverse ratio of the lengths, but with a sensible deficiency in the longer pieces.

Experiments 5 and 7 show the strength to be proportional to the breadth and the square of the depth.

Experiments 1 and 7 show the same thing, compounded with the inverse ratio of the length; the deficiency of which is not so remarkable here.

Experiments 1 and 2, and experiments 5 and 6, show the increase of strength, by fastening down the ends, to be in the proportion of 2 to 3; which the theory states as 2 to 4, the difference being probably owing to the manner of fixing.

Mr. Buffon made numerous experiments, both on small bars, and on large ones, which are the best. The following is a specimen of one set, made on bars of sound oak, clear of knots.

Length. feet.	Weight. lbs.	Broke with lbs.	Bent. inch.	Time. min.
7	60	5350	3·5	29′
	56	5275	4·5	22
8	68	4600	3·75	15
	63	4500	4·7	13
9	77	4100	4·85	14
	71	3950	5·5	12
10	84	3625	5·83	15
	82	3600	6·5	15
12	100	3050	7	
	98	2925	8	

Column 1 shows the length of the bar, in feet, clear between the supports.—Column 2 is the weight of the bar in lbs, the 2d day after it was felled.—Column 3 shows the number of pounds necessary for breaking the tree in a few minutes.—Col. 4 is the number of inches it bent down before breaking.—Col. 5 is the time at which it broke.—The parts next the root were always the heaviest and strongest.

The following experiments on other sizes were made in the same way, two at least of each length being taken; and the table contains the mean results. The beams were all squared, and their sides in inches are placed at the top of the columns, their

their lengths in feet being in the first column. The numbers in the other columns, are the pounds weight which broke the pieces.

	4	5	6	7	8	A
7	5312	11525	18950	32200	47649	11525
8	4550	9787	15525	26050	39750	10085
9	4025	8308	13150	22350	32800	8964
10	3612	7125	11250	19475	27750	8068
12	2987	6075	9100	16175	23450	6723
14		5300	7475	13225	19775	5763
16		4350	6362	11000	16375	5042
18		3700	5562	9245	13200	4482
20		3225	4950	8375	11487	4034
22		2975				3667
24		2162				3362
28		1775				2881

Mr. Buffon had found, by many trials, that oak timber lost much of its strength in the course of seasoning or drying; and therefore, to secure uniformity, his trees were all felled in the same season of the year, were squared the day after, and the experiments tried the 3d day. Trying them in this green state gave him an opportunity of observing a very curious phenomenon. When the weights were laid quickly on, nearly sufficient to break the beam, a very sensible smoke was observed to issue from the two ends with a sharp hissing sound; which continued all the time the tree was bending and cracking. This shows the great effects of the compression, and that the beam is strained through its whole length, which is shown also by its bending through the whole length.

Mr. Buffon considers the experiments with the 5-inch bars as the standard of comparison, having both extended these to greater lengths, and also tried more pieces of each length. Now, the theory determines the relative strength of bars, of the same section, to be inversely as their lengths: but most of the trials show a great deviation from this rule, probably owing, in part at least, to the weights of the pieces themselves. Thus, the 5-inch bar of 28 feet long should have half the strength of that of 14 feet, or 2650, whereas it is only 1775; the bar of 14 feet should have half the strength of that of 7 feet, or 5762, but is only 5300; and so of others. The column A is added, to show the strength that each of the 5-inch bars ought to have by the theory.

Mr.

Mr. Banks, an ingenious lecturer on natural philosophy, has made many experiments on the strength of oak, deal, and iron. He found that the worst or weakest piece of dry heart of oak, 1 inch square, and 1 foot long, broke with 602 lbs, and the strongest piece with 974 lbs; the worst piece of deal broke with 464 lbs, and the best with 690 lbs. A like bar of the worst kind of cast iron 2190 lbs. Bars of iron set up in positions oblique to the horizon, showed strengths nearly proportional to the sines of elevation of the pieces. Equal bars placed horizontally, on supports 3 feet distant, bore $6\frac{3}{4}$ cwt; the same at $2\frac{1}{2}$ feet distance broke only with 9 cwt.—An arched rib of $29\frac{1}{2}$ feet span, and 11 inches high in the centre, supported $99\frac{1}{2}$ cwt; it sunk in the middle $3\frac{7}{8}$ inches, and rose again $\frac{3}{4}$ on removing the load. The same rib tried without abutments, broke with 55 cwt.—Another rib, a segment of a circle, $29\frac{1}{4}$ feet span, and 3 feet high in the middle, bore $100\frac{1}{2}$ cwt, and sunk $1\frac{3}{16}$ in the middle. The same rib without abutments, broke with $64\frac{1}{2}$ cwt.

Mr. Banks made also experiments at another foundry, on like bars of 1 inch square, each yard in length weighing 9 lbs, the props at 3 feet asunder.

The 1st bar broke with	963 lbs.
The 2d ditto	958
The 3d ditto	994
Bar made from the cupola, broke with . . .	864
Bar equally thick in the middle, but the ends shaped into a parabola, and weighed $6\frac{3}{16}$ lbs, broke with	874

From these and many other experiments, Mr. Banks concludes, that cast iron is from $3\frac{1}{3}$ to $4\frac{1}{2}$ times stronger than oak of the same dimensions, and from 5 to $6\frac{1}{2}$ times stronger than deal.

Some Examples for Practice.

The theory, as has been before mentioned, is, That the strength of a bar, or the weight it will bear, is directly as the breadth and square of the depth divided by the length. So that, if b denote the breadth of a bar, d the depth, l the length, and w the weight it will bear; and the capitals B, D, L, W denote the like quantities in another bar; then, by the rule $\frac{bd^2}{l} : w :: \frac{BD^2}{L} : W$, which gives this general equation $bd^2LW = BD^2lw$, from which any one of the letters is easily found, when the rest are given.

Now, if we take, for a standard of comparison, this experiment of Mr. Banks, that a bar of oak an inch square and a

foot

foot in length, lying on a prop at each end, and its strength, or the utmost weight it can bear, on its middle, 660 lbs: here $b=1$, $d=1$, $l=1$, $w=660$; these substituted in the above equation, it becomes $LW = 660BD^2$, from which any one of the four quantities L, W, B, D, may be found, when the other three are given, when the calculation respects oak timber. But for fir the like rule will be $LW = 440BD^2$; and for iron $LW = 2640BD^2$.

Exam. 1. Required the utmost strength of an oak beam, of 6 inches square and 8 feet long, supported at each end, or the weight to break it in the middle?

Here are given $B = 6$, $D = 6$, $L = 8$, to find $W = \frac{660BD^2}{L} = \frac{660 \times 6 \times 36}{8} = 660 \times 3 \times 9 = 17820$ lbs.

Exam. 2. Required the depth of an oak beam, of the same length and strength as above, but only 3 inches breadth?

Here, as $3 : 6 :: 36 : D^2 = 72$, theref. $D = \sqrt{72} = 8{\cdot}485$ the depth.

This last beam, though as strong as the former, is but little more than $\frac{2}{3}$ of its size or quantity. And thus, by making joists thinner, a great part of the expense is saved, as in the modern style of flooring, &c.

Exam. 3. To determine the utmost strength of a deal joist of 2 inches thick and 8 inches deep, the bearing or breadth of the room being 12 feet?—Here $B = 2$, $D = 8$, $L = 12$; then the rule $LW = 440BD^2$ gives $W = \frac{440 \times B \times D^2}{L} = \frac{440 \times 2 \times 64}{12} = \frac{440 \times 32}{3} = 4693$ lbs.

Exam. 4. Required the depth of a bar of iron 2 inches broad and 8 feet long, to sustain a load of 20,000lbs?—Here $B = 2$, $L = 8$, and $W = 20{,}000$, to find D from the equation $LW = 2640BD^2$, viz, $D^2 = \frac{LW}{2640B} = \frac{8 \times 20000}{2640 \times 2} = \frac{1000}{33} = 30{\cdot}3$, and $D = \sqrt{30{\cdot}3} = 5\frac{1}{2}$ inches, the depth.

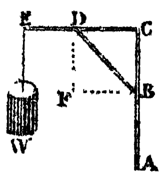
Exam. 5. To find the length of a bar of oak, an inch square, so that when supported at both ends it may just break by its own weight?—Here, according to the notation and calculation in prob. 36, $l = 1$, $w = \frac{2}{5}$ of a lb, the weight of 1 foot in length, and $u = 660$ lbs. Then $L = l\sqrt{\frac{w+2u}{w}} = \sqrt{3301} = 57{\cdot}45$ feet, nearly.

Exam. 6. To find the length of an iron bar an inch square, that it may break by its own weight, when it is supported at both

both ends.—Here as before $l = 1$, $w = 3$ lbs nearly the weight of 1 foot in length, also $u = 2640$. Therefore $L = l\sqrt{\frac{w+2u}{w}} = 41\cdot97$ feet nearly.

Note. It might perhaps have been supposed that this last result should exceed the preceding one: but it must be considered that while iron is only about 4 times stronger than oak, it is at least 8 times heavier.

Exam. 7. When a weight W is suspended from E on the arm of a crane ABCDE, it is required to find the pressure at the end D of the spur, and that at B against the upright post AC.

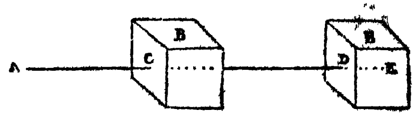


Here, by the nature of the lever, $\frac{CE}{CD}W =$ the pressure at D in the vertical direction DF: but this pressure in DF is to that in DB as DF to DB, viz, DF : DB :: $\frac{CE}{CD}W : \frac{CE \cdot DB}{DF \cdot CD}W$ the pressure in DB; and again, DB : FB or CD :: $\frac{CE \cdot DB}{DF \cdot CD}W : \frac{CE}{DF}W = \frac{CE}{BC}W$ the pressure against B in direction FB.

Thus, for example, if CE $= 16$ feet, BC $= 6$, CD $= 8$, BD $= 10$, and W $= 3$ tons: then $\frac{CE \cdot BD}{BC \cdot CD}W = \frac{16 \cdot 10}{6 \cdot 8} \times 3 = 10$ tons, for the pressure on the spur DB. Also $\frac{CE}{CB}W = \frac{16}{6} \times 3 = 8$ tons, the force tending to break the bar AC at B.

PROBLEM 41.

To determine the circumstances of Space, Penetration, Velocity, and Time, arising from a Ball moving with a Given Velocity, and striking a Moveable Block of Wood, or other substance.



Let the ball move in the direction AE passing through the centre of gravity of the block B, impinging on the point C; and when the block has moved through the space CD in consequence of the blow, let the ball have penetrated to the depth DE.

Let

Let B = the mass or matter in the block,
b = the same in the ball,
s = CD the space moved by the block,
x = DE the penetration of the ball, and theref.
$s + x$ = CE the space described by the ball,
a = the first velocity of the ball,
v = the velocity of the ball at E,
u = velocity of the block at the same instant,
t = the time of penetration, or of the motion,
r = the resisting force of the wood.

Then shall $\frac{r}{B}$ be the accelerating force of the block,

and $\frac{r}{b}$ the retarding force of the ball.

Now because the momentum $B\dot{u}$, communicated to the block in the time $\dot{t}$, is that which is lost by the ball, namely, $-b\dot{v}$, therefore $B\dot{u} = -b\dot{v}$, and $Bu = -bv$. But when $v = a$, $u = 0$; therefore, by correcting, $Bu = b(a - v)$; or the momentum of the block is every where equal to the momentum lost by the ball. And when the ball has penetrated to the utmost depth, or when $u = v$, this becomes $Bu = b(a - u)$, or $ab = (B + b)u$; that is, the momentum before the stroke, is equal to the momentum after it. And the velocity communicated will be the same, whatever be the resisting force of the block, the weight being the same.

Again, by theor. 6, Forces, vol. 2, it is $u^2 = \frac{4grs}{B}$, and $-v^2 = \frac{4gr}{b} \times (s + x)$, or rather, by correction, $a^2 - v^2 = \frac{4gr}{b}(s + x)$. Hence the penetration or $x = \frac{b(a^2 - v^2) - 4grs}{4gr}$. And when $v = u$, by substituting u for v, and Bu^2 for $4grs$, the greatest penetration becomes $\frac{ba^2 - (B + b)u^2}{4gr}$; and this again, by writing ab for its value $(B + b)u$, gives the greatest penetration $x = \frac{Bba^2}{4gr(B + b)} = \frac{ba^2}{4gr} \times (1 - \frac{b}{B + b})$. Which is barely equal to $\frac{ba^2}{4gr}$ when the block is fixed, or infinitely great; and is always very nearly equal to the same $\frac{ba^2}{4gr}$ when B is very great in respect of b.

Hence $s + x = \frac{a^2 - u^2}{4gr}b = \frac{a^2 - \frac{a^2b^2}{(B + b)^2}}{4gr}b = \frac{B^2 + 2Bb}{(B + b)^2} \times \frac{a^2b}{4gr}$.

And

And theref. $B + b : B + 2b :: x : s + x$, or $B + b : b :: x : s$, and $s = \frac{bx}{B + b} = \frac{Bb^2a^2}{4gr(B+b)^2}$.

Exam. When the ball is iron, and weighs 1 pound, it penetrates elm about 13 inches when it moves with a velocity of 1500 feet per second, in which case,

$$\frac{r}{b} = \frac{a^2}{4gx} = \frac{1500^2}{4 \times 16\frac{1}{12} \times \frac{13}{12}} = \frac{9000^2}{193 \times 13} = 32284 \text{ nearly.}$$

When $B = 500$lb, and $b = 1$; then $u = \frac{ab}{B+b} = \frac{1500}{501} = 3$ feet nearly per second, the velocity of the block.

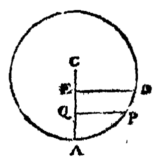
Also $s = \frac{Bu^2}{4gr} = \frac{500 \times 9}{4 \times 16\frac{1}{12} \times 32284} = \frac{1}{461\frac{1}{2}}$ part of a foot, or $\frac{2}{77}$ of an inch, which is the space moved by the block when the ball has completed its penetration.

And $t = \frac{2s}{u} = \frac{2}{461\frac{1}{2} \times 3} = \frac{1}{692}$ part of a second, or

$t = \frac{2s + 2x}{v} = \frac{\frac{2}{461\frac{1}{2}} + \frac{26}{12}}{1500} = \frac{6 + 13 \,.\, 231}{6.231 \,.\, 1500} = \frac{1}{692}$ part of a second, the time of penetration.

PROBLEM 42.

To find the Velocity and Time of a Heavy Body descending down the Arc of a Circle, or vibrating in the Arc by a Line fixed in the Centre.



Let D be the beginning of the descent, C the centre, and A the lowest point of the circle; draw DE and PQ perpendicular to AC. Then the velocity in P being the same as in Q by falling through EQ, it will be $v = 2\sqrt{(g \times EQ)} = 8\sqrt{(a - x)}$, when $a = AE$, $x = AQ$.

But the flux. of the time $\dot{t}$ is $= \frac{-\dot{AP}}{v}$, and $\dot{AP} = \frac{r\dot{x}}{\sqrt{(2rx - x^2)}}$ where $r =$ the radius AC. Theref. $\dot{t} = \frac{r}{8} \times \frac{-\dot{x}}{\sqrt{(2rx-x^2)} \times \sqrt{(d-x)}}$ $= \frac{d}{16} \times \frac{-\dot{x}}{\sqrt{(ax - x^2)} \times \sqrt{(d - x)}} = \frac{-\sqrt{d}}{16} \times \frac{\dot{x}}{\sqrt{(ax - x^2)} \times \sqrt{(1 - \frac{x}{d})}}$, where $d = 2r$ the diameter.

Or $\dot{t} = \frac{-\sqrt{d}}{16} \times \frac{\dot{x}}{\sqrt{(ax - x^2)}} \left(1 + \frac{x}{2d} + \frac{1 \,.\, 3x^2}{2 \,.\, 4d^2} + \frac{1 \,.\, 3 \,.\, 5x^3}{2 \,.\, 4 \,.\, 6d^3} \text{ \&c}\right)$, by developing $1 \div \sqrt{(1 - \frac{x}{d})}$, or $(1 - \frac{x}{d})^{-\frac{1}{2}}$, in a series.

But

But the fluent of $\frac{\dot{x}}{\sqrt{(ax-x^2)}}$ is $\frac{2}{a} \times$ arc to radius $\frac{1}{2}a$ and vers. x, or it is the arc whose rad. is 1 and vers. $\frac{2x}{a}$: which call A. And let the fluents of the succeeding terms, without the coefficients, be B, C, D, E, &c. Then will the flux. of any one, as $\dot{Q}$, at n distance from A, be $\dot{Q} = x^n \dot{A} = x\dot{P}$, which suppose also = the flux. of $bP - dx^{n-1}\sqrt{(ax - x^2)} = b\dot{P} - d(n-1)\dot{x}x^{n-2}\sqrt{(ax - x^2)} - d\dot{x}x^{n-2} \times \frac{\frac{1}{2}ax - x^2}{\sqrt{(ax - x^2)}} = b\dot{P} - d\dot{x} \times \frac{(n-\frac{1}{2})ax^{n-1} - nx^n}{\sqrt{(ax-x^2)}} = b\dot{P} - d(n - \frac{1}{2})a\dot{P} + dnx\dot{P}$.

Hence, by equating the coefficients of the like terms, $d = \frac{1}{n}$; $b = \frac{2n-1}{2n}a$; and $Q = \frac{(2n-1)aP - 2x^{n-1}\sqrt{(ax - x^2)}}{2n}$.
Which being substituted, the fluential terms become $\frac{\sqrt{d}}{16} \times$ $(- A - \frac{1}{2d} \cdot \frac{aA - 2\sqrt{(ax - x^2)}}{2} - \frac{1 \,.\, 3}{2 \,.\, 4d^2} \cdot \frac{3aB - 2x\sqrt{(ax - x^2)}}{4} - \frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6d^3} \cdot \frac{5aC - 2x^2\sqrt{(ax-x^2)}}{6} - \&c)$. Or the same fluents will be found by art. 32 pa. 238.

But when $x = a$, those terms become barely $\frac{3 \cdot 1416\sqrt{d}}{16} \times$ $(-1 - \frac{1^2a}{2^2d} - \frac{1^2 \,.\, 3^2a^2}{2^2 \,.\, 4^2a^2} - \frac{1^2 \,.\, 3^2 \,.\, 5^2a^3}{2^2 \,.\, 4^2 \,.\, 6^2d^3} - \&c)$; which being subtracted, and x taken = 0, there arises for the whole time of descending down DA, or the corrected value of $t =$ $\frac{3 \cdot 1416\sqrt{d}}{16} \times (1 + \frac{1^2a}{2^2d} + \frac{1^2 \,.\, 3^2a^2}{2^2 \,.\, 4^2d^2} + \frac{1^2 \,.\, 3^2 \,.\, 5^2a^3}{2^2 \,.\, 4^2 \,.\, 6^2d^3} + \&c)$.

When the arc is small, as in the vibration of the pendulum of a clock, all the terms of the series may be omitted after the second, and then the time of a semi-vibration t is nearly $= \frac{1 \cdot 5708}{4}\sqrt{\frac{r}{2}} \times (1 + \frac{a}{8r})$. And theref. the times of vibration of a pendulum, in different arcs, are as $8r + a$, or 8 times the radius added to the versed sine of the arc.

If D be the degrees of the pendulum's vibration, on each side of the lowest point of the small arc, the radius being r, the diameter d, and $3 \cdot 1416 = p$; then is the length of that arc $A = \frac{prD}{180} = \frac{pdD}{360}$. But the versed sine in terms of the arc is $a = \frac{A^2}{2r} - \frac{A^4}{24r^3} + \&c = \frac{A^2}{d} - \frac{A^4}{3d^3} + \&c$. Therefore $\frac{a}{d} = \frac{A^2}{d^2} - \frac{A^4}{3d^4} + \&c = \frac{p^2D^2}{360^2} - \frac{p^4D^4}{3 \,.\, 360^4} + \&c$, or only $= \frac{p^2D^2}{360^2}$ the

the first term, by rejecting all the rest of the terms on account of their smallness, or $\frac{a}{d} = \frac{a}{2r}$ nearly $= \frac{D^2}{13131}$. This value then being substituted for $\frac{a}{d}$ or $\frac{a}{2r}$ in the last near value of the time, it becomes $t = \frac{1\cdot5708}{4}\sqrt{\frac{r}{2}} \times (1 + \frac{D^2}{52524})$ nearly. And therefore the times of vibration in different small arcs, are as $52524 + D^2$, or as 52524 added to the square of the number of degrees in the arc.

Hence it follows that the time lost in each second, by vibrating in a circle, instead of the cycloid, is $\frac{D^2}{52524}$; and consequently the time lost in a whole day of 24 hours, or $24 \times 60 \times 60$ seconds, is $\frac{5}{3}D^2$ nearly. In like manner, the seconds lost per day by vibrating in the arc of Δ degrees, is $\frac{5}{3}\Delta^2$. Therefore, if the pendulum keep true time in one of these arcs, the seconds lost or gained per day, by vibrating in the other, will be $\frac{5}{3}(D^2 - \Delta^2)$. So, for example, if a pendulum measure true time in an arc of 3 degrees, it will lose $11\frac{2}{3}$ seconds a day by vibrating 4 degrees; and $26\frac{2}{3}$ seconds a day by vibrating 5 degrees; and so on.

And in like manner, we might proceed for any other curve, as the ellipse, hyperbola, parabola, &c.

Scholium. By comparing this with the results of the problems 13 and 14 in vol. 2, it will appear that the times in the cycloid, and in the arc of a circle, and in any chord of the circle, are respectively as the three quantities

$$1,\ 1 + \frac{a}{8r}\ \&c, \text{ and } \frac{1}{\cdot7854},$$

or nearly as the three quantities $1,\ 1 + \frac{a}{8r},\ 1\cdot27324$; the first and last being constant, but the middle one, or the time in the circle, varying with the extent of the arc of vibration. Also the time in the cycloid is the least, but in the chord the greatest; for the greatest value of the series, in this prob. when $a = r$, or the arc AD is a quadrant, is $1\cdot18014$; and in that case the proportion of the three times is as the numbers 1, 1·18014, 1·27324. Moreover the time in the circle approaches to that in the cycloid, as the arc decreases, and they are very nearly equal when that arc is very small.

PROBLEM

PROBLEM 43.

To find the Time and Velocity of a Chain, consisting of very small links, descending from a smooth horizontal plane; the Chain being 100 inches long, and 1 inch of it hanging off the Plane at the Commencement of Motion.

Put a = 1 inch, the length at the beginning;
l = 100 the whole length of the chain;
x = any variable length off the plane.

Then x is the motive force to move the body,

and $\frac{x}{l} = f$ the accelerative force.

Hence $v\dot{v} = 2gf\dot{s} = 2g \times \frac{x}{l} \times \dot{x} = \frac{2gx\dot{x}}{l}$.

The fluents give $v^2 = \frac{2gx^2}{l}$. But $v = 0$ when $x = a$, theref. by correction, $v^2 = 2g \times \frac{x^2 - a^2}{l}$, and $v = \sqrt{(2g \times \frac{x^2 - a^2}{l})}$ the velocity for any length x. And when the chain just quits the plain, $x = l$, and then the greatest velocity is $\sqrt{(2g \times \frac{l^2 - a^2}{l})} = \sqrt{(2 \times 193 \times \frac{100^2 - 1^2}{100})} = \sqrt{\frac{386 \times 9999}{100}} =$ 196·45902 inches, or 16·371585 feet, per second.

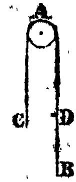
Again $\dot{t}$ or $\frac{\dot{s}}{v} = \sqrt{\frac{l}{2g}} \times \frac{\dot{x}}{\sqrt{(x^2 - a^2)}}$; the correct fluent of which is $t = \sqrt{\frac{l}{2g}} \times \log. \frac{x + \sqrt{(x^2 - a^2)}}{a}$, the time for any length x. And when $x = l = 100$, it is $t = \sqrt{\frac{100}{386}} \times \log. \frac{100 + \sqrt{9999}}{1} = 2·69676$ seconds, the time when the last of the chain just quits the plane.

PROBLEM 44.

To find the Time and Velocity of a Chain, of very small Links, quitting a Pulley, by passing freely over it: the whole Length being 200 Inches, and the one End hanging 2 Inches below the other at the Beginning.

Put $a = 2$, $l = 200$, and x = BD any variable difference of the two parts AB, AC. Then $\frac{x}{l} = f$, and $v\dot{v}$ or $2gf\dot{s} = 2g \cdot \frac{x}{l} \cdot \frac{1}{2}\dot{x} = \frac{gx\dot{x}}{l}$. Hence the correct fluent is $v^2 = g \times \frac{x^2 - a^2}{l}$, and $v = \sqrt{(g \times \frac{x^2 - a^2}{l})}$, the general expression for the

veloc.



veloc. And when $x = l$, or when C arrives at A, it is $v = \sqrt{(g \times \frac{l^2 - a^2}{l})} = \sqrt{(193 \times \frac{200^2 - 2^2}{200})} = \sqrt{(386 \times \frac{100^2 - 1^2}{100})} = \sqrt{\frac{386 \times 9999}{100}} = 196\cdot45902$ inches, or $16\cdot371585$ feet, for the greatest velocity when the chain just quits the pulley.

Again, $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{\dot{x}}{2v} = \sqrt{\frac{l}{4g}} \times \frac{\dot{x}}{\sqrt{(x^2 - a^2)}}$. And the correct fluent is $t = \sqrt{\frac{l}{4g}} \times \log. \frac{x + \sqrt{(x^2 - a^2)}}{a}$, the general expression for the time. And when $x = l$, it becomes $t = \sqrt{\frac{l}{4g}} \times \log. \frac{l + \sqrt{(l^2 - a^2)}}{a} = \sqrt{\frac{200}{772}} \times \log. \frac{200 + \sqrt{(200^2 - 2^2)}}{2} = \sqrt{\frac{100}{386}} \times \log. \frac{100 + \sqrt{9999}}{1} = 2\cdot69676$ seconds, the whole time when the chain just quits the pulley.

So that the velocity and time at quitting the pulley in this prob. and the plane in the last prob. are the same; the distance descended 99 being the same in both. For, though the weight l moved in this latter case, be double of what it was in the former, the moving force x is also double, because here the one end of the chain shortens as much as the other end lengthens, so that the space descended $\frac{1}{2}x$ is doubled, and becomes x; and hence the accelerative force $\frac{x}{l}$ or f is the same in both; and of course the velocity and time the same for the same distance descended.

PROBLEM 45.

To find the Number of Vibrations made by two Weights, connected by a very fine Thread, passing freely over a Tack or a Pulley, while the less Weight is drawn up to it by the Descent of the heavier Weight at the other End; the Extent of the vibrations being Indefinitely Small.

Suppose the motion to commence at equal distances below the pulley at B; and that the weights are 1 and 2 pounds.

Put a = AB, half the length of the thread;
b = $39\frac{1}{8}$ inc. or $3\frac{25}{96}$ feet, the second's pend.
x = Bw = BW, any space passed over;
z = the number of vibrations.

Then $\frac{W - w}{W + w} = f = \frac{1}{3}$ is the accelerating force.

And hence v or $\sqrt{4gfs} = \sqrt{4gfx}$, and $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{4gfx}}$.

But,

But, by the nature of pendulums, $\sqrt{(a \pm x)} : \sqrt{b} :: 1$ vibr. : $\sqrt{\frac{b}{a \pm x}}$ the vibrations per second made by either weight, namely, the longer or shorter, according as the upper or under sign is used, if the threads were to continue of that length for 1 second. Hence, then, as

$$1'' : \dot{t} :: \sqrt{\frac{b}{a \pm x}} : \dot{z} = \dot{t}\sqrt{\frac{b}{a \pm x}} = \sqrt{\frac{b}{4gf}} \times \frac{\dot{x}}{\sqrt{(ax \pm x^2)}},$$

the fluxion of the number of vibrations.

Now when the upper sign + takes place, the fluent is $z = 2\sqrt{\frac{b}{4gf}} \times 1. \frac{\sqrt{x} + \sqrt{(a+x)}}{\sqrt{a}} = \sqrt{\frac{b}{gf}} \times 1. \frac{ax + \sqrt{(ax + x^2)}}{a}$. And when $x = a$, the same then becomes $z = \sqrt{\frac{b}{gf}} \times \log.\ 1 + \sqrt{2} = \sqrt{\frac{3b}{g}} \times \log.\ 1 + \sqrt{2} = \sqrt{\frac{117\frac{3}{8}}{193}} \times \log.\ 1 + \sqrt{2} =$ ·688511, the whole number of vibrations made by the descending weight.

But when the lower sign, or −, takes place, the fluent is $\sqrt{\frac{b}{4gf}} \times$ arc to rad. 1 and vers. $\frac{2x}{a}$. Which, when $x = a$, gives $\frac{1}{2}p\sqrt{\frac{b}{gf}} = 3{\cdot}1416 \times \sqrt{\frac{3 \times 39\frac{1}{8}}{4 \times 193}} = \frac{3.1416}{2} \times \sqrt{\frac{117\frac{3}{8}}{193}} =$ 1·227091, the whole number of vibrations made by the lesser or ascending weight.

Schol. It is evident that the whole number of vibrations, in each case, is the same, whatever the length of the thread is. And that the greater number is to the less, as 1·5708 to the hyp log. of $1 + \sqrt{2}$.

Farther, the number of vibrations performed in the same time t, by an invariable pendulum, constantly of the same length a, is $\sqrt{\frac{b}{gf}} =$ ·781190. For, the time of descending the space a, or the fluent of $\dot{t} = \frac{\dot{x}}{\sqrt{4gfx}}$, when $x = a$, is $t = \sqrt{\frac{a}{gf}}$. And, by the nature of pendulums, $\sqrt{a} : \sqrt{b} ::$ 1 vibr. : $\sqrt{\frac{b}{a}}$ the number of vibrations performed in 1 second; hence $1'' : t :: \sqrt{\frac{b}{a}} : t\sqrt{\frac{b}{a}} = \sqrt{\frac{b}{gf}}$, the constant number of vibrations.

So that the three numbers of vibrations, namely, of the ascending, constant, and descending pendulums, are proportional to the numbers 1·5708, 1, and hyp. log. $1 + \sqrt{2}$, or as 1·5708, 1, and ·88137; whatever be the length of the thread.

PROBLEM

PROBLEM 46.

To determine the Circumstances of the Ascent and Descent of two unequal Weights, suspended at the two Ends of a Thread, passing over a Pulley: the Weight of the Thread and of the Pulley being considered in the Solution.

Let $l =$ the whole length of the thread;
$a =$ the weight of the same;
$b =$ Aw the dif. of lengths at first;
$d =$ W − w the dif. of the two weights;
$e =$ a weight applied to the circumference, such as to be equal to its whole wt. and friction reduced to the circumference;
$s =$ W $+ w + a + c$ the sum of the weights moved.

Then the weight of b is $\frac{ab}{l}$, and $d - \frac{ab}{l}$ is the moving force at first. But if x denote any variable space descended by W, or ascended by w, the difference of the lengths of the thread will be altered $2x$; so that the difference will then be $b - 2x$, and its weight $\frac{b-2x}{l}a$; conseq. the motive force there will be $d - \frac{b-2x}{l}a = \frac{dl-ab+2ax}{l}$, and theref. $\frac{dl-ab+2ax}{sl} = f$ the accelerating force there. Hence then $v\dot{v} = 2gf\dot{x} = 2g\dot{x} \times \frac{dl-ab+2ax}{sl}$; the fluents of which give $v^2 = 4gx \times \frac{dl-ab+ax}{sl}$, or $v = 2\sqrt{\frac{ag}{sl}} \times \sqrt{(ex+x^2)}$ the general expression for the velocity, putting $e = \frac{dl-ab}{a}$. And when $x = b$, or W becomes as far below w as it was above it at the beginning, it is barely $v = 2\sqrt{\frac{bdg}{s}}$ for the velocity at that time. Also, when a, the weight of the thread, is nothing, the velocity is only $2\sqrt{\frac{dgx}{s}}$, as it ought.

Again, for the time, $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{1}{2}\sqrt{\frac{sl}{ag}} \times \frac{\dot{x}}{\sqrt{(ex+x^2)}}$; the fluents of which give $t = \sqrt{\frac{sl}{ag}} \times \log. \frac{\sqrt{x}+\sqrt{(e+x)}}{\sqrt{e}}$ the general expression for the time of descending any space x.

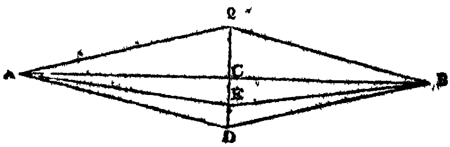
And if the radicals be expanded in a series, and the log. of it be taken, the same will become

$$t = \sqrt{\frac{sx}{dg}} \times \sqrt{\frac{dl}{dl-ab}} \times \left(1 - \frac{x}{6e} + \frac{3x^2}{40e^2} \text{ \&c}\right).$$

Which therefore becomes barely $\sqrt{\frac{sx}{dg}}$ when a, the weight of the thread, is nothing; as it ought.

PROBLEM 47.

To find the Velocity and Time of Vibration of a small Weight, fixed to the middle of a Line, or fine Thread void of Gravity, and stretched by a given Tension; the Extent of the Vibration being very small.



Let l = AC half the length of the thread;
a = CD the extent of the vibration;
x = CE any variable distance from C;
w = wt. of the small body fixed to the middle;
W = a wt. which, hung at each end of the thread, will be equal to the constant tension at each end, acting in the direction of the thread.

Now, by the nature of forces, AE : CE : : W the force in direction EA : the force in direction EC. Or, because AC is nearly = AE, the vibration being very small, taking AC instead of AE, it is AC : CE : : W : $\frac{Wx}{l}$ the force in FC arising from the tension in EA. Which will be also the same for that in EB. Therefore the sum is $\frac{2Wx}{l}$ = the whole motive force in FC arising from the tensions on both sides. Consequently $\frac{2Wx}{lw} = f$ the accelerative force there. Hence the equation of the fluxions $v\dot{v}$ or $2gf\dot{s} = \frac{-4gWx\dot{x}}{lw}$; and the flus. $v^2 = -\frac{4gWx^2}{lw}$. But when $x = a$, this is $-\frac{4gWa^2}{lw}$, and should be = 0; theref. the correct fluents are $v^2 = 4gW \times \frac{a^2 - x^2}{lw}$, and hence $v = \sqrt{(4gW \times \frac{a^2 - x^2}{lw})}$ the velocity of the little body w at any point E. And when $x = 0$, it is $v = 2a\sqrt{\frac{gW}{lw}}$ for the greatest velocity at the point C.

Now if we suppose w = 1 grain, W = 5lb troy, or 28800 grains, and $2l$ = AB = 3 feet; the velocity at C becomes $a\sqrt{\frac{8 \times 16\frac{1}{12} \times 28800}{3}} = 1111\frac{2}{5}a$. So that

if $a = \frac{1}{10}$ inc. the greatest veloc. is $9\frac{1}{15}$ ft. per sec.
if a = 1 inc. the greatest veloc. is $92\frac{37}{60}$ ft. per sec.
if a = 6 inc. the greatest veloc. is $555\frac{7}{10}$ ft. per sec.

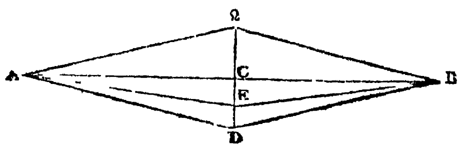
To

To find the time t, it is $\dot{t}$ or $\frac{-\dot{r}}{v} = \frac{1}{2}\sqrt{\frac{lw}{wg}} \times \frac{-\dot{x}}{\sqrt{(a^2 - x^2)}}$. Hence the correct fluent is $t = \frac{1}{2}\sqrt{\frac{wl}{wg}} \times$ arc to cosine $\frac{x}{a}$ and radius 1, for the time in DE. And when $x = 0$, the whole time in DC, or of half a vibration, is $\cdot 7854\sqrt{\frac{wl}{wg}}$; and conseq. the time of a whole vibration through Dd is $1\cdot 5708\sqrt{\frac{wl}{wg}}$.

Using the foregoing numbers, namely $w = 1$, w = 28800, and $2l = 3$ feet; this expression for the time gives $\frac{1111\frac{3}{5}}{3\cdot 1416} = 353\frac{5}{9}$, the number of vibrations per second. But if $w = 2$, there would be 250 vibrations per second; and if $w = 100$, there would be $35\frac{16}{45}$ vibrations per second.

PROBLEM 48.

To determine the same as in the last Problem, when the Distance CD *bears some sensible Proportion to the Length* AB; *the Tension of the Thread however being still supposed a Constant Quantity.*



Using here the same notation as in the last problem, and taking the true variable length AE for AC, it is AE or EB : CE :: $2\text{w} : \frac{2\text{w}x}{\text{AE}} = \frac{2\text{w}x}{\sqrt{(l^2 + x^2)}}$ the whole motive force from the two equal tensions w in AE and EB; and theref. $\frac{2\text{w}}{w} \times \frac{x}{\sqrt{(l^2 + x^2)}} = f$ is the accelerative force at E. Theref. the fluxional equation is $v\dot{v}$ or $2gf\dot{s} = \frac{4\text{w}g}{w} \times \frac{-x\dot{x}}{\sqrt{(l^2 + x^2)}}$; and the fluents $v^2 = \frac{8\text{w}g}{w} \times -\sqrt{(l^2 + x^2)}$. But when $x = a$, these are $0 = \frac{8\text{w}g}{w} \times -\sqrt{(l^2 + a^2)}$; therefore the correct fluents are $v^2 = \frac{8\text{w}g}{w} \times [\sqrt{(l^2 + a^2)} - \sqrt{(l^2 + x^2)}] = \frac{8\text{w}g}{w} \times (\text{AD} - \text{AE})$. And hence $v = \sqrt{[\frac{8\text{w}g}{w} \times (\text{AD} - \text{AE})]}$ the general expression for the velocity at E. And when E arrives at C, it gives the greatest

greatest velocity there $= \sqrt{[\frac{8wg}{w} \times (\text{AD} - \text{AC})]}$. Which, when $\text{W} = 28800$, $w = 1$, $2l = 3$ feet, and $\text{CD} = 6$ inches or $\frac{1}{2}$ a foot, is $\sqrt{(8 \times 28800 \times 16\frac{1}{12} \times \frac{\sqrt{10} - 3}{2})} = 548\frac{1}{3}$ feet per second. Which came out $555\frac{7}{10}$ in the last problem, by using always AC for AE in the value of f. But when the extent of the vibrations is very small, as $\frac{1}{10}$ of an inch, as it commonly is, this greatest velocity here will be $\sqrt{8} \times 28800 \times 16\frac{1}{12} \times \frac{1}{115200} = 9\frac{1}{4}$ nearly, which in the last problem was $9\frac{5}{12}$ nearly.

To find the time, it is $\dot{t}$ or $\frac{-\dot{x}}{v} = \sqrt{\frac{w}{8wg}} \times \frac{-\dot{x}}{\sqrt{[c - \sqrt{(l^2 + x^2)}]}}$, making $c = \text{AD} = \sqrt{(l^2 + a^2)}$. To find the fluent the easier, multiply the numer. and denom. both by $\sqrt{[c + \sqrt{(l^2 + x^2)}]}$, so shall $\dot{t} = \sqrt{\frac{w}{8wg}} \times \frac{-\dot{x}}{\sqrt{(a^2 - x^2)}} \times \sqrt{[c + \sqrt{(l^2 + x^2)}]}$. Expand now the quantity $\sqrt{[c + \sqrt{(l^2 + x^2)}]}$ in a series, and put $d = c + l$, so shall $\dot{t} = \sqrt{\frac{wd}{8wg}} \times \frac{-\dot{x}}{\sqrt{(a^2 - x^2)}} (1 + \frac{x^2}{4dl} - \frac{2d + l}{32d^2l^3}x^4 + \frac{4d^2 + 2dl + l^2}{128d^3l^5}x^6 - \frac{40d^3 + 8d^2l + 12dl^2 + 5l^3}{2048d^4l^7}x^8$ &c). Now the fluent of the first term $\frac{\dot{x}}{\sqrt{(a^2 - x^2)}}$ is $=$ the arc to sine $\frac{x}{a}$ and radius 1, which arc call A; and let P, Q be the fluents of any other two successive terms, without the coefficients, the distance of Q from the first term A being n; then it is evident that $\dot{\text{Q}} = x^2\dot{\text{P}} = x^{2n}\dot{\text{A}}$, and $\dot{\text{P}} = x^{2n-2}\dot{\text{A}}$. Assume theref. $\text{Q} = b\text{P} - ex^{2n-1}\sqrt{(a^2 - x^2)}$; then is $\dot{\text{Q}}$ or $x^2\dot{\text{P}} = b\dot{\text{P}} - (2n - 1)ex^{2n-2}\dot{x}\sqrt{(a^2 - x^2)} + \frac{ex^{2n}\dot{x}}{\sqrt{(a^2 - x^2)}} = b\dot{\text{P}} - \frac{(2n - 1)ea^2x^{2n-2}\dot{x}}{\sqrt{(a^2 - x^2)}} + \frac{(2n-1)ex^{2n}\dot{x}}{\sqrt{(a^2 - x^2)}} + \frac{ex^{2n}\dot{x}}{\sqrt{(a^2 - x^2)}} = b\dot{\text{P}} - (2n - 1)ea^2\dot{\text{P}} + (2n - 1)ex^2\dot{\text{P}} + ex^2\dot{\text{P}} = b\dot{\text{P}} - (2n - 1)ea^2\dot{\text{P}} + 2nex^2\dot{\text{P}}$. Then comparing the coefficients of the like terms, we find $1 = 2en$, and $b = (2n - 1)ea^2$; from which are obtained $e = \frac{1}{2n}$, and $b = \frac{2n-1}{2n}a^2$. Consequently $\text{Q} = \frac{(2n - 1)a^2\text{P} - x^{2n-1}\sqrt{(a^2 - x^2)}}{2n}$, the general equation between any two successive terms, and by means of which the series may be continued as far as we please. And hence, neglecting the coefficients, putting $\text{A} =$ the first term, namely the arc whose sine is $\frac{x}{a}$, and B, C, D, &c, the following

ing terms, the series is as follows, $A + \frac{a^2A - x\sqrt{(a^2 - x^2)}}{2} + \frac{3a^2B - x^3\sqrt{(a^2 - x^2)}}{4} + \frac{5a^2C - x^5\sqrt{(a^2 - x^2)}}{6}$ &c. Now when $x = 0$, this series $= 0$; and when $x = a$, the series becomes $\frac{1}{4}p + \frac{a^2A}{2} + \frac{3a^2B}{4} + \frac{5a^2C}{6}$ &c, where $p = 3{\cdot}1416$, or the series is $\frac{1}{4}p(1 + \frac{1}{2}a^2 + \frac{1 \cdot 3}{2 \cdot 4}a^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}a^6$ &c.)

So that, by taking in the coefficients, the general time of passing over any distance DE will be

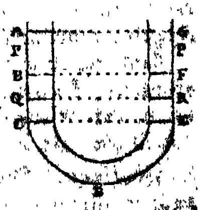
$\sqrt{\frac{w(c+l)}{8wg}} \times \frac{1}{2}p \times (1 + \frac{1}{4dl} \cdot \frac{1}{2}a^2 - \frac{2d+l}{32d^2l^3} \cdot \frac{1 \cdot 3}{2 \cdot 4}a^4$ &c, $-$ arc sin. $\frac{x}{a} - \frac{1}{4dl} \cdot \frac{a^2A - x\sqrt{(a^2 - x^2)}}{2} + \frac{2d+l}{32d^2l^3} \cdot \frac{3a^2B - x^3\sqrt{(a^2 - x^2)}}{4}$ &c.

And hence, taking $x = 0$, and doubling, the time of a whole vibration, or double the time of passing over CD will be equal to $\frac{1}{2}p\sqrt{\frac{w(c+l)}{2wg}} \times (1 + \frac{1}{4dl} \cdot \frac{1}{2}a^2 - \frac{2d+l}{32d^2l^3} \cdot \frac{1 \cdot 3}{2 \cdot 4}a^4 + \frac{4d^2+2dl+l^2}{128d^3l^5} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}a^6 - \frac{40d^3+8d^2l+12dl^2+5l^3}{2048d^4l^7} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}a^8$ &c.) Which, when $a = 0$, or $c = l$, becomes only $\frac{1}{2}p\sqrt{\frac{wl}{wg}}$, the same as in the last problem, as it ought.

Taking here the same numbers as in the last problem, viz, $l = \frac{3}{4}$, $a = \frac{1}{4}$, $w = 2$, $W = 28800$, $g = 16\frac{1}{12}$; then $\frac{1}{2}p\sqrt{\frac{w(c+l)}{2wg}} = {\cdot}0040514$, and the series is $1 + {\cdot}006762 - {\cdot}000175 + {\cdot}000003$ &c $= 1{\cdot}006590$; therefore ${\cdot}0040514 \times 1{\cdot}006590 = {\cdot}0040965 = \frac{1}{245\frac{1}{4}}$ is the time of one whole vibration, and consequently $245\frac{1}{4}$ vibrations are performed in a second; which were 250 in the last problem.

PROBLEM 49.

It is proposed to determine the Velocity, and the Time of Vibration, of a Fluid in the Arms of a Canal or bent Tube.



Let the tube ABCDEF have its two branches AC, GE vertical, and the lower part CDE in any position whatever, the whole being of a uniform diameter or width throughout. Let water, or quicksilver, or any other fluid, be poured in, till it stand in equilibrio, at any hori- zontal

zontal line EF. Then let one surface be pressed or pushed down by shaking, from E to C, and the other will ascend through the equal space FG; after which let them be permitted freely to return. The surfaces will then continually vibrate in equal times between AC and EG. The velocity and times of which oscillations are therefore required.

When the surfaces are any where out of a horizontal line, as at P and Q, the parts of the fluid in QDR, on each side, below QR, will balance each other; and the weight of the part in PR, which is equal to 2PF, gives motion to the whole. So that the weight of the part 2PF is the motive force by which the whole fluid is urged, and therefore $\frac{\text{wt. of } 2\text{PF}}{\text{whole wt.}}$ is the accelerative force. Which weights being proportional to their lengths, if l be the length of the whole fluid, or axis of the tube filled, and a = FG or BC; then is $\frac{2a}{l}$ the accelerative force. Putting theref. x = GP any variable distance, v the velocity, and t the time; then PF $= a - x$, and $\frac{2a - 2x}{l} = f$ the accelerative force; hence $v\dot{v}$ or $2gf\dot{s} = \frac{4g}{l}(a\dot{x} - x\dot{x})$; the fluents of which give $v^2 = \frac{4g}{l}(2ax - x^2)$, and $v = \sqrt{(4g \times \frac{2ax - x^2}{l})}$ is the general expression for the velocity at any term. And when $x = a$, it becomes $v = 2a\sqrt{\frac{g}{l}}$ for the greatest velocity at B and F.

Again, for the time, we have $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{1}{2}\sqrt{\frac{l}{g}} \times \frac{\dot{x}}{\sqrt{(2ax - x^2)}}$, the fluents of which give $t = \frac{1}{2}\sqrt{\frac{l}{g}} \times$ arc to versed sine $\frac{x}{a}$ and radius 1, the general expression for the time. And when $x = a$, it becomes $t = \frac{1}{4}p\sqrt{\frac{l}{g}}$ for the time of moving from G to F, p being $= 3{\cdot}1416$; and consequently $\frac{1}{2}p\sqrt{\frac{l}{g}}$ the time of a whole vibration from G to E, or from C to A. And which therefore is the same, whatever AB is, the whole length l remaining the same.

And the time of vibration is also equal to the time of the vibration of a pendulum whose length is $\frac{1}{2}l$, or half the length of the axis of the fluid. So that, if the length l be $78\frac{1}{4}$ inches, it will oscillate in 1 second.

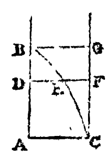
Scholium. This reciprocation of the water in the canal, is nearly similar to the motion of the waves of the sea. For the

the time of vibration is the same, however short the branches are, provided the whole length be the same. So that when the height is small, in proportion to the length of the canal, the motion is similar to that of a wave, from the top to the bottom or hollow, and from the bottom to the top of the next wave; being equal to two vibrations of the canal; the whole length of a wave, from top to top, being double the length of the canal. Hence the wave will move forward by a space nearly equal to its breadth, in the time of two vibrations of a pendulum whose length is ($\frac{1}{2}l$) half the length of the canal, or one-fourth the breadth of a wave, or in the time of one vibration of a pendulum whose length is the whole breadth of the wave, since the times of vibration are as the square roots of their lengths. Consequently, waves whose breadth is equal to $39\frac{1}{8}$ inches, or $3\frac{25}{96}$ feet, will move over $3\frac{25}{96}$ feet in a second, or $195\frac{5}{8}$ feet in a minute, or nearly 2 miles and a quarter in an hour. And the velocity of greater or less waves will be increased or diminished in the subduplicate ratio of their breadths.

Thus, for instance, for a wave of 18 inches breadth, as $\sqrt{39\frac{1}{8}} : 39\frac{1}{8} :: \sqrt{18} : \sqrt{(39\frac{1}{8} \times 18)} = \frac{3}{2}\sqrt{313} = 26{\cdot}5377$ the velocity of the wave of 18 inches breadth.

PROBLEM 50.

To determine the Time of emptying any Ditch, or Inundation, &c, by a Cut or Notch, from the Top to the Bottom of it.



Let x = AB the variable height of water at any time;
b = AC the breadth of the cut;
d = the whole or first depth of water;
A = the area of the surface of the water in the ditch;
$g = 16\frac{1}{12}$ feet.

The velocity at any point D, is as $\sqrt{\text{BD}}$, that is, as the ordinate DE of a parabola BEC, whose base is AC, and altitude AB. Therefore the velocities at all the points in AB, are as all the ordinates of the parabola. Consequently the quantity of water running through the cut ABGC, in any time, is to the quantity which would run through an equal aperture placed all at the bottom in the same time, as the area of the parabola ABC, to the area of the parallelogram ABGC, that is, as 2 to 3.

But $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity at AC; therefore $\frac{2}{3} \times 2\sqrt{gx} \times bx = \frac{4}{3}bx\sqrt{gx}$ is the quantity discharged per second

second through ABGC; and consequently $\frac{4bx\sqrt{gx}}{3A}$ is the velocity per second of the descending surface. Hence then $\frac{4bx\sqrt{gx}}{3A} : -\dot{x} :: 1'' : \frac{-3A\dot{x}}{4bx\sqrt{gx}} = \dot{t}$ the fluxion of the time of descending.

Now when A the surface of the water is constant, or the ditch is equally broad throughout, the correct fluent of this fluxion gives $t = \frac{3A}{2b\sqrt{g}} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{dx}}$ for the general time of sinking the surface to any depth x. And when $x = 0$, this expression is infinite; which shows that the time of a complete exhaustion is infinite.

But if $d = 9$ feet, $b = 2$ feet, $A = 21 \times 1000 = 21000$, and it be required to exhaust the water down to $\frac{1}{16}$ of a foot deep; then $x = \frac{1}{16}$, and the above expression becomes $\frac{3 \times 21000}{4 \times 4\frac{1}{16}} \times \frac{3 - \frac{1}{4}}{\frac{3}{4}} = 14400''$, or just 4 hours for that time. And if it be required to depress it 8 feet, or till 1 foot depth of water remain in the ditch, the time of sinking the water to that point will be $43' \, 38''$.

Again, if the ditch be the same depth and length as before, but 20 feet broad at bottom, and 22 at top; then the descending surface will be a variable quantity, and, by prob. 16 vol. 2, it will be $\frac{90 + x}{90} \times 20000$; hence in this case the flux. of the time, or $\frac{-3A\dot{x}}{4bx\sqrt{gx}}$, becomes $\frac{-500}{3b\sqrt{g}} \times \frac{90 + x}{x\sqrt{x}}\dot{x}$; the correct fluent of which is $t = \frac{1000}{3b\sqrt{g}} \times (\frac{90 - x}{\sqrt{x}} - \frac{90 - d}{\sqrt{d}})$ for the time of sinking the water to any depth x.

Now when $x = 0$, this expression for the complete exhaustion becomes infinite.

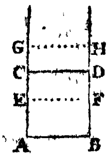
But if . . $x = 1$ foot, the time t is $42' \, 56''\frac{1}{2}$.

And when $x = \frac{1}{16}$ foot, the time is $3^h \, 50' \, 28''\frac{1}{2}$.

PROBLEM 51.

To determine the Time of filling the Ditches of a Fortification 6 Feet deep with Water, through the Sluice of a Trunk of 3 Feet Square, the Bottom of which is level with the Bottom of the Ditch, and the Height of the supplying Water is 9 Feet above the Bottom of the Ditch.

Let ACDB represent the area of the vertical sluice, being a square of 9 square feet, and AB level with the bottom of the ditch. And suppose the ditch filled to any height AE, the surface being then at EF.



Put $a = 9$ the height of the head or supply;
$b = 3 =$ AB $=$ AC;
$g = 16\frac{1}{12}$;
A $=$ the area of a horizontal section of the ditches;
$x = a -$ AE, the height of the head above EF.

Then $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity with which the water presses through the part AEFB; and theref. $2\sqrt{gx} \times$ AEFB $= 2b\sqrt{gx}(a - x)$ is the quantity per second running through AEFB. Also, the quantity running per second through ECDF is $2\sqrt{gx} \times \frac{11}{12}$ECDF $= \frac{11}{6}b\sqrt{gx}(b - a + x)$ nearly. For the real quantity is, by proceeding as in the last prob. the difference between two parab. segs. the alt. of the one being x, its base b, and the alt. of the other $a - b$; and the medium of that dif. between its greatest state at AB, where it is $\frac{2}{15}$AD, and its least state at CD, where it is 0, is nearly $\frac{11}{12}$ED. Consequently the sum of the two, or $\frac{1}{6}b\sqrt{gx}$ $(a + 11b - x)$ is the quantity per second running in by the whole sluice ACDB. Hence then $\frac{1}{6}b\sqrt{gx} \times \frac{a + 11b - x}{A} = v$ is the rate or velocity per second with which the water rises in the ditches; and so $v : -\dot{x} :: 1'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-6A}{b\sqrt{g}} \times \frac{x^{-\frac{1}{2}}\dot{x}}{c - x}$ the fluxion of the time of filling to any height AE, putting $c = a + 11b$.

Now when the ditches are of equal width throughout, A is a constant quantity, and in that case the correct fluent of this fluxion is $t = \frac{6A}{b\sqrt{gc}} \times \log. (\frac{\sqrt{c} + \sqrt{a}}{\sqrt{c} - \sqrt{a}} \times \frac{\sqrt{c} - \sqrt{x}}{\sqrt{c} + \sqrt{x}})$ the general expression for the time of filling to any height AE, or $a - x$, not exceeding the height AC of the sluice. And when $x =$ AC $= a - b = d$ suppose, then $t = \frac{6A}{b\sqrt{gc}} \times \log.$ $(\frac{\sqrt{c} + \sqrt{a}}{\sqrt{c} - \sqrt{a}} . \frac{\sqrt{c} - \sqrt{d}}{\sqrt{c} + \sqrt{d}})$ is the time of filling to CD the top of the sluice.

Again, for filling to any height GH above the sluice, x denoting as before $a -$ AG the height of the head above GH, $2\sqrt{gx}$ will be the velocity of the water through the whole sluice AD: and therefore $2b^2\sqrt{gx}$ the quantity per second, and $\frac{2b^2\sqrt{gx}}{A} = v$ the rise per second of the water in the ditches; consequently $v : -\dot{x} :: 1'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-A}{2b^2\sqrt{g}} \times \frac{\dot{x}}{\sqrt{x}}$ the general fluxion of the time; the correct fluent of which, being

being 0 when $x = a - b = d$, is $t = \frac{A}{b^2\sqrt{g}}(\sqrt{d} - \sqrt{x})$ the time of filling from CD to GH.

Then the sum of the two times, namely, that of filling from AB to CD, and that of filling from CD to GH, is $\frac{A}{b\sqrt{g}}\left[\frac{\sqrt{d}-\sqrt{x}}{b} + \frac{6}{\sqrt{c}}\log.\left(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}}\cdot\frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}+\sqrt{d}}\right)\right]$ for the whole time required. And, using the numbers in the prob., this becomes $\frac{A}{3\sqrt{g}}\left[\frac{\sqrt{6}-\sqrt{3}}{3} + \frac{6}{\sqrt{42}} \times 1.\left(\frac{\sqrt{42}+\sqrt{9}}{\sqrt{42}-\sqrt{9}}\cdot\frac{\sqrt{42}-\sqrt{6}}{\sqrt{42}+\sqrt{6}}\right)\right]$ $= 0{\cdot}03577277A$, the time in terms of A the area of the length and breadth, or horizontal section of the ditches. And if we suppose that area to be 200000 square feet, the time required will be $7154''$, or $1^h\ 59'\ 14''$.

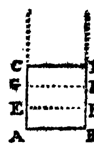
And if the sides of the ditch slope a little, so as to be a little narrower at the bottom than at top, the process will be nearly the same, substituting for A its variable value, as in the preceding problem. And the time of filling will be very nearly the same as that above determined.

PROBLEM 52.

But if the Water, from which the Ditches are to be filled, be the Tide, which at Low Water is below the Bottom of the Trunk, and rises to 9 Feet above the Bottom of it by a regular Rise of One Foot in Half an Hour; it is required to ascertain the Time of Filling it to 6 Feet high, as before in the Last Problem.

Let ACDB represent the sluice; and when the tide has risen to any height GH, below CD the top of the sluice, without the ditches, let EF be the mean height of the water within. And put $b = 3 = AB = AC$;

$g = 16\frac{1}{12}$;
A = horizontal section of the ditches;
$x = AG$;
$z = AE$.



Then $\sqrt{g} : \sqrt{EG} :: 2g : 2\sqrt{g(x-z)}$ the velocity of the water through AEFB; and
$\sqrt{g} : \sqrt{EG} :: \frac{4}{3}g : \frac{4}{3}\sqrt{g(x-z)}$ the mean vel. through EGHF;
theref. $2bz\sqrt{g(x-z)}$ is the quantity per sec. through AEFB;
and $\frac{4}{3}b(x-z)\sqrt{g(x-z)}$ is the same through EGHF;
conseq. $\frac{2}{3}b\sqrt{g} \times (2x+z)\sqrt{(x-z)}$ is the whole through AGHB per second. This quantity divided by the surface A, gives $\frac{2b\sqrt{g}}{3A} \times (2x+z)\sqrt{(x-z)} = v$ the velocity per second

 with

with which EF, or the surface of the water in the ditches, rises. Therefore

$$v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{3A}{2b\sqrt{g}} \times \frac{\dot{z}}{(2x+z)\sqrt{(x-z)}}.$$

But, as GH rises uniformly 1 foot in 30′ or 1800″, therefore 1 : AG :: 1800″ : 1800x = t the time of the tide rising through AG; conseq. $\dot{t} = 1800\dot{x} = \frac{3A}{2b\sqrt{g}} \times \frac{\dot{z}}{(2x+z)\sqrt{(x-z)}}$, or $m\dot{z} = (2x+z)\sqrt{(x-z)}\,.\,\dot{x}$ is the fluxional equa. expressing the relation between x and z; where $m = \frac{A}{1200b\sqrt{g}} = \frac{3200}{231}$ or $13\frac{197}{231}$ when A = 200000 square feet.

Now to find the fluent of this equation, assume $z = Ax^{\frac{5}{2}} + Bx^{\frac{8}{2}} + Cx^{\frac{11}{2}} + Dx^{\frac{14}{2}}$ &c. So shall

$$\sqrt{(x-z)} = x^{\frac{1}{2}} - \frac{A}{2}x^{\frac{4}{2}} - \frac{A^2+4B}{8}x^{\frac{7}{2}} - \frac{A^3+4AB+8C}{16}x^{\frac{10}{2}} \text{ \&c,}$$

$$2x + z = 2x + Ax^{\frac{5}{2}} + Bx^{\frac{8}{2}} + Cx^{\frac{11}{2}} \text{ \&c,}$$

$$(2x+z)\sqrt{(x-z)}\dot{x} = 2x^{\frac{3}{2}}\dot{x} * - \frac{3A^2}{4}x^{\frac{9}{2}}\dot{x} - \frac{A^3+6AB}{4}x^{\frac{12}{2}}\dot{x} \text{ \&c,}$$

$$\text{and } m\dot{z} = \tfrac{5}{2}mAx^{\frac{3}{2}}\dot{x} + \tfrac{8}{2}mBx^{\frac{6}{2}}\dot{x} + \tfrac{11}{2}mCx^{\frac{9}{2}}\dot{x} + \tfrac{14}{2}mDx^{\frac{12}{2}}\dot{x} \text{ \&c.}$$

Then equate the coefficients of the like terms,

so shall	and consequently
$\frac{5}{2}mA = 2,$	$A = \frac{4}{5m},$
$\frac{3}{2}mB = 0,$	$B = 0,$
$\frac{11}{2}mC = -\frac{3}{4}A^2,$	$C = -\frac{24}{275m^3},$
$\frac{14}{2}mD = -\frac{3}{4}A^3 - \frac{3}{2}AB,$	$D = -\frac{16}{875m^4},$
&c;	&c.

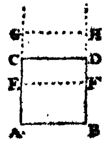
Which values of A, B, C, &c, substituted in the assumed value of z, give

$$z = \frac{4}{5m}x^{\frac{5}{2}} * - \frac{24}{275m^3}x^{\frac{11}{2}} - \frac{16}{875m^4}x^{\frac{14}{2}} \text{ \&c;}$$

$$\text{or } z = \frac{4}{5m}x^{\frac{5}{2}} \text{ very nearly.}$$

And when $x = 3$ = AC, then z = ·886 of a foot, or $10\frac{5}{8}$ inches, = AE, the height of the water in the ditches when the tide is at CD or 3 feet high without, or in the first hour and half of time.

Again,



Again, to find the time, after the above, when EF arrives at CD, or when the water in the ditches arrives as high as the top of the sluice.

The notation remaining as before, then $2bz\sqrt{g(x-z)}$ per sec. runs through AF, and $\frac{2}{5}b(3-z)\sqrt{g(x-z)}$ per sec. thro' ED nearly; therefore $\frac{2}{5}b\sqrt{g}\times(12+z)\sqrt{(x-z)}$ is the whole per second through AD nearly.

conseq. $\frac{2b\sqrt{g}}{5A}\times(12+z)\sqrt{(x-z)}=v$ is the velocity per second of the point E; and therefore

$$v : \dot{z} :: 1'' : \dot{t}=\frac{\dot{z}}{v}=\frac{5A}{2b\sqrt{g}}\times\frac{\dot{z}}{(12+z)\sqrt{(x-z)}}=1800\dot{x}, \text{ or}$$

$m\dot{z}=(12+z)\sqrt{(x-z)}\,.\,\dot{x}$, where $m=\frac{A}{720b\sqrt{g}}=23\frac{2}{7}$ nearly.

Assume $z=Ax^{\frac{3}{2}}+Bx^{\frac{4}{2}}+Cx^{\frac{5}{2}}+Dx^{\frac{6}{2}}$ &c. So shall

$$\sqrt{(x-z)}=x^{\frac{1}{2}}-\frac{A}{2}x^{\frac{2}{2}}-\frac{A^2+4B}{8}x^{\frac{3}{2}}-\frac{A^3+4AB+8C}{16}x^{\frac{4}{2}} \text{ \&c};$$

$$12+z=12+Ax^{\frac{3}{2}}+Bx^{\frac{4}{2}}+Cx^{\frac{6}{2}} \text{ \&c};$$

$$(12+z)\,.\,\sqrt{(x-z)}\,.\,\dot{x}=12x^{\frac{1}{2}}\dot{x}-6Ax^{\frac{2}{2}}\dot{x}-(\tfrac{3}{2}A^2+6B)x^{\frac{3}{2}}\dot{x} \text{ \&c};$$

$$m\dot{z}=\tfrac{3}{2}mAx^{\frac{1}{2}}\dot{x}+\tfrac{4}{2}mBx^{\frac{2}{2}}\dot{x}+\tfrac{5}{2}mCx^{\frac{3}{2}}\dot{x} \text{ \&c},$$

Then, equating the like terms, &c, we have

$$A=\frac{8}{m},\ B=-\frac{24}{m^2},\ C=\frac{96}{5m^3},\ D=\frac{64}{3m^4} \text{ nearly, \&c.}$$

Hence $z=\frac{8}{m}x^{\frac{3}{2}}-\frac{24}{m^2}x^2+\frac{96}{5m^3}x^{\frac{5}{2}}+\frac{64}{3m^4}x^3$ &c.

Or $z=\frac{8}{m}x^{\frac{3}{2}}$ nearly.

But, by the first process, when $x=3$, $z=\cdot886$; which substituted for them, we have $z=\cdot886$, and the series $=1\cdot63$; therefore the correct fluents are

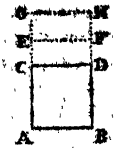
$$z-\cdot886=-1\cdot63+\frac{8}{m}x^{\frac{3}{2}}-\frac{24}{m^2}x^2 \text{ \&c},$$

$$\text{or } z+\cdot744=\frac{8}{m}x^{\frac{3}{2}}-\frac{24}{m^2}x^2 \text{ \&c.}$$

And when $z=3=$ AC, it gives $x=6\cdot369$ for the height of the tide without, when the ditches are filled to the top of the sluice, or 3 feet high; which answers to $3^h\ 11'\ 4''$.

Lastly, to find the time of rising the remaining 3 feet above the top of the sluice; let

$x=$ CG



$x = $ CG the height of the tide above CD,
$z = $ CE ditto in the ditches above CD;
and the other dimensions as before.
Then $\sqrt{g} : \sqrt{\text{EG}} :: 2\sqrt{g} : 2\sqrt{g(x-z)} =$ the velocity with which the water runs through the whole sluice AD; conseq. AD $\times\ 2\sqrt{g(x-z)} = 18\sqrt{g(x-z)}$ is the quantity per second running through the sluice, and $\frac{18\sqrt{g}}{\text{A}}\sqrt{(x-z)} = v$ the velocity of z, or the rise of the water in the ditches, per second; hence $v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{\text{A}}{18\sqrt{g}} \times \frac{\dot{z}}{\sqrt{(x-z)}} = 1800\dot{x}$, and $m\dot{z} = \dot{x}\sqrt{(x-z)}$ is the fluxional equation; where $m = \frac{\text{A}}{180^2\sqrt{g}} = \frac{3200}{2079}$.

To find the fluent,

Assume $z = \text{A}x^{\frac{3}{2}} + \text{B}x^{\frac{4}{2}} + \text{C}x^{\frac{5}{2}} + \text{D}x^{\frac{6}{2}}$ &c.

Then $x - z = x - \text{A}x^{\frac{3}{2}} - \text{B}x^{\frac{4}{2}} - \text{C}x^{\frac{5}{2}}$ &c,

$$\dot{x}\sqrt{(x-z)} = x^{\frac{1}{2}}\dot{x} - \frac{\text{A}}{2}x^{\frac{2}{2}}\dot{x} - \frac{\text{A}^2+4\text{B}}{8}x^{\frac{3}{2}}\dot{x} \text{ \&c.}$$

$$m\dot{z} = \tfrac{3}{2}n\text{A}x^{\frac{1}{2}}\dot{x} + \tfrac{4}{2}n\text{B}x^{\frac{2}{2}}\dot{x} + \tfrac{5}{2}n\text{C}x^{\frac{3}{2}}\dot{x} \text{ \&c.}$$

Then equating the like terms gives

$$\text{A} = \frac{2}{3n},\ \text{B} = \frac{-1}{6n^2},\ \text{C} = \frac{1}{90n^3},\ \text{D} = \frac{-1}{810n^4}, \text{ \&c.}$$

Hence $z = \frac{2}{3n}x^{\frac{3}{2}} - \frac{1}{6n^2}x^2 + \frac{1}{90n^3}x^{\frac{5}{2}} - \frac{1}{810n^4}x^3$ &c.

But, by the second case, when $z = 0$, $x = 3{\cdot}369$, which being used in the series, it is $1{\cdot}936$; therefore the correct fluent is $z = -1{\cdot}936 + \frac{2}{3n}x^{\frac{3}{2}} - \frac{1}{6n^2}x^2$ &c. And when $z = 3$, $x = 7$; the heights above the top of the sluice, answering to 6 and 10 feet above the bottom of the ditches. That is, for the water to rise to the height of 6 feet within the ditches, it is necessary for the tide to rise to 10 feet without, which just answers to 5 hours; and so long it would take to fill the ditches 6 feet deep with water, their horizontal area being 200000 square feet.

Further, when $x = 6$, then $z = 2{\cdot}117$ the height above the top of the sluice; to which add 3, the height of the sluice, and the sum $5{\cdot}117$, is the depth of water in the ditches in 4 hours and a half, or when the tide has risen to the height of 9 feet without the ditches.

Note. In the foregoing problems, concerning the efflux of

of water, it is taken for granted that the velocity is the same as that which is due to the whole height of the surface of the supplying water: a supposition which agrees with the principles of the greater number of authors: though some make the velocity to be that which is due to the half height only; and others make it still less.

Also in some places, where the difference between two parabolic segments was to be taken, in estimating the mean velocity of the water through a variable orifice, I have used a near mean value of the expression; which makes the operation of finding the fluents much more easy, and is at the same time sufficiently exact for the purpose in hand.

We may further add a remark here concerning the method of finding the fluents of the three fluxional forms that occur in the solution of this problem, viz, the three forms $m\dot{z} = (2x + z)\sqrt{(x - z)}\dot{x}$, and $m\dot{z} = (12 + z)\sqrt{(x - z)}\dot{x}$, and $m\dot{z} = \sqrt{(x - z)}\dot{x}$, the fluents of which are found by assuming the fluent mz in an infinite series ascending in terms of x with indeterminate coefficients A, B, C, &c, which coefficients are afterwards determined in the usual way, by equating the corresponding terms of two similar and equal series, the one series denoting one side of the fluxional equation, and the other series the other side. By similar series, is meant when they have equal or like exponents; though it is not necessary that the exponents of all the terms should be like or pairs, but only some of them, as those that are not in pairs will be cancelled or expelled by making their coefficients = 0 or nothing. Now the general way to make the two series similar, is to assume the fluent z equal to a series in terms of x, either ascending or descending, as here

$z = x^r + x^{r+s} + x^{r+2s}$ &c for ascending,

or $z = x^r + x^{r-s} + x^{r-2s}$ &c for a descending

series, having the exponents r, $r \pm s$, $r \pm 2s$, &c in arithmetical progres ion, the first term r, and common difference s; without the general coefficients A, B, C, &c, till the values of the exponents be determined. In terms of this assumed series for z, find the values of the two sides of the given fluxional equation, by substituting in it the said series instead of z; then put the exponent of the first term of the one side equal that of the other, which will give the value of the first exponent r; in like manner put the exponents of the two 2d terms equal, which will give the value of the common difference s; and hence the whole series of exponents r, $r \pm s$, $r \pm 2s$, &c, becomes known.

Thus, for the last of the three fluxional equations above mentioned, viz, $m\dot{z} = \sqrt{(x - z)}\dot{x}$, or only $\dot{z} = \sqrt{(x - z)}\dot{x}$; having

having assumed as above $z = x^r + x^{r+s}$ &c, and taking the fluxion, then $\dot{z} = x^{r-1}\dot{x} + x^{r+s-1}\dot{x} +$ &c, omitting the coefficients; and the other side of the equation $\sqrt{(x-z)}\dot{x} = \sqrt{(x - x^r - x^{r+s}}$ &c$) = x^{\frac{1}{2}}\dot{x} - x^{r-\frac{1}{2}}\dot{x}$ &c. Now the exponents of the first terms made equal, give $r - 1 = \frac{1}{2}$, theref. $r = 1 + \frac{1}{2} = \frac{3}{2}$; and those of the 2d terms made equal, give $r+s-1 = r-\frac{1}{2}$, theref. $s-1 = -\frac{1}{2}$, and $s = 1 - \frac{1}{2} = \frac{1}{2}$; conseq. the whole assumed series of exponents r, $r + s$, $r + 2s$, &c, become $\frac{3}{2}, \frac{4}{2}, \frac{5}{2}$, &c, as assumed above in pa. 374.

Again, for the 2d equation $m\dot{z}$ or $\dot{z} = (12 + z)\sqrt{(x-z)}\dot{x} = (a+z)\sqrt{(x - z)}\dot{x}$; assuming $z = x^r + x^{r+s}$ &c as before, then $\dot{z} = x^{r-1}\dot{x} + x^{r+s-1}\dot{x}$ &c, and $\sqrt{(x-z)}\dot{x} = x^{\frac{1}{2}}\dot{x} - x^{r-\frac{1}{2}}\dot{x}$ &c, both as above; this mult. by $a + z$ or $a + x^r + x^{r+s}$ &c, gives $ax^{\frac{1}{2}}\dot{x} - ax^{r-\frac{1}{2}}\dot{x}$ &c: then equating the first exponents gives $r - 1 = \frac{1}{2}$ or $r = \frac{3}{2}$, and $r + s - 1 = r - \frac{1}{2}$, or $s = 1 - \frac{1}{2} = \frac{1}{2}$; hence the series of exponents is $\frac{3}{2}, \frac{4}{2}, \frac{5}{2}$, &c, the same as the former, and as assumed in pa. 373.

Lastly, assuming the same form of series for z and $\dot{z}$ as in the above two cases, for the 1st fluxional equation also, viz, $m\dot{z} = (2x + z)\sqrt{(x-z)}\dot{x}$: then $\sqrt{(x-z)}\dot{x} = x^{\frac{1}{2}}\dot{x} - x^{r-\frac{1}{2}}\dot{x}$ &c, which mult. by $2x + z$, gives $2x^{\frac{3}{2}}\dot{x} - x^{r+\frac{1}{2}}\dot{x}$ &c: here equating the first exponents gives $r - 1 = \frac{3}{2}$ or $r = \frac{5}{2}$, and equating the 2d exponents gives $r + s - 1 = r + \frac{1}{2}$, or $s = \frac{3}{2}$; hence the series of exponents in this case is $\frac{5}{2}, \frac{8}{2}, \frac{11}{2}$, &c, as used for this case in pa. 372. Then, in every case, the general coefficients A, B, C, &c, are joined to the assumed terms x^r, x^{r+s}, &c, and the whole process conducted as in the three pages just referred to.

Such then is the regular and legitimate way of proceeding, to obtain the form of the series with respect to the exponents of the terms. But, in many cases we may perceive at sight, without that formal process, what the law of the exponents will be, as I indeed did in the solutions in the pages above referred to; and any person with a little practice may easily do the same.

PROBLEM 53.

To determine the fall of the Water in the Arches of a Bridge.

The effects of obstacles placed in a current of water, such as the piers of a bridge, are, a sudden steep descent, and an increase of velocity in the stream of water, just under the arches, more or less in proportion to the quantity of the obstruction and velocity of the current: being very small and hardly perceptible where the arches are large and the piers few

few or small, but in a high and extraordinary degree at London-bridge, and some others, where the piers and the sterlings are so very large, in proportion to the arches. This is the case, not only in such streams as run always the same way, but in tide rivers also, both upward and downward, but much less in the former than in the latter. During the time of flood, when the tide is flowing upward, the rise of the water is against the under side of the piers; but the difference between the two sides gradually diminishes as the tide flows less rapidly towards the conclusion of the flood. When this has attained its full height, and there is no longer any current, but a stillness prevails in the water for a short time, the surface assumes an equal level, both above and below bridge. But, as soon as the tide begins to ebb or return again, the resistance of the piers against the stream, and the contraction of the waterway, cause a rise of the surface above and under the arches, with a fall and a more rapid descent in the contracted stream just below. The quantity of this rise, and of the consequent velocity below, keep both gradually increasing, as the tide continues ebbing, till at quite low water, when the stream or natural current being the quickest, the fall under the arches is the greatest. And it is the quantity of this fall which it is the object of this problem to determine.

Now, the motion of free running water is the consequence of, and produced by the force of gravity, as well as that of any other falling body. Hence the height due to the velocity, that is, the height to be freely fallen by any body to acquire the observed velocity of the natural stream, in the river a little way above bridge, becomes known. From the same velocity also will be found that of the increased current in the narrowed way of the arches, by taking it in the reciprocal proportion of the breadth of the river above, to the contracted way in the arches; viz, by saying, as the latter is to the former, so is the first velocity, or slower motion, to the quicker. Next, from this last velocity, will be found the height due to it as before, that is, the height to be freely fallen through by gravity, to produce it. Then the difference of these two heights, thus freely fallen by gravity, to produce the two velocities, is the required quantity of the waterfall in the arches; allowing however, in the calculation, for the contraction, in the narrowed passage, at the rate as observed by Sir I. Newton, in prop. 36 of the 2d book of the Principia, or by other authors, being nearly in the ratio of 25 to 21. Such then are the elements and principles on which the solution of the problem is easily made out as follows.

Let

Let b = the breadth of the channel in feet;
v = mean velocity of the water in feet per second;
c = breadth of the waterway between the obstacles.

Now $25 : 21 :: c : \frac{21}{25}c$, the waterway contracted as above.

And $\frac{21}{25}c : b :: v : \frac{25b}{21c}v$, the velocity in the contracted way.

Also $32^2 : v^2 :: 16 : \frac{1}{64}v^2$, height fallen to gain the velocity v.

And $32^2 : (\frac{25b}{21c}v)^2 :: 16 : (\frac{25b}{21c})^2 \times \frac{1}{64}v^2$, ditto for the vel. $\frac{25b}{21c}v$.

Then $(\frac{25b}{21c})^2 \times \frac{v^2}{64} - \frac{v^2}{64}$ is the measure of the fall required.

Or $[(\frac{25b}{21c})^2 - 1] \times \frac{vv}{64}$ is a rule for computing the fall.

Or rather $\frac{1 \cdot 42b^2 - c^2}{64c^2} \times v^2$ very nearly, for the fall.

EXAM. 1. *For London-bridge.*

By the observations made by Mr. Labelye in 1746,
The breadth of the Thames at London-bridge is 926 feet;
The sum of the waterways at the time of low-water is 236 ft;
Mean velocity of the stream just above bridge is $3\frac{1}{6}$ ft. per sec.
But under almost all the arches are driven into the bed great numbers of what are called dripshot piles, to prevent the bed from being washed away by the fall. These dripshot piles still further contract the waterways, at least $\frac{1}{6}$ of their measured breadth, or near 39 feet in the whole; so that the waterway will be reduced to 197 feet, or in round numbers suppose 200 feet.

Then $b = 926$, $c = 200$, $v = 3\frac{1}{6} = \frac{19}{6}$.

Hence $\frac{1 \cdot 44b^2 - c^2}{64c^2} = \frac{1217616 - 40000}{64 \times 40000} = \cdot 46$.

And $v^2 = \frac{19^2}{6^2} = 10\frac{1}{36}$.

Theref. $\cdot 46 \times 10\frac{1}{36} = 4 \cdot 63$ ft. $=$ 4 ft. $7\frac{1}{2}$ in. the fall required.

By the most exact observations made about the year 1736, the measure of the fall was 4 feet 9 inches.

EXAM. 2. *For Westminster-bridge.*

Though the breadth of the river at Westminster-bridge is 1220 feet; yet, at the time of the greatest fall, there is water through only the 13 large arches, which amount to but 820 feet; to which adding the breadth of the 12 intermediate piers, equal to 174 feet, gives 994 for the breadth of the river

river at that time; and the velocity of the water a little above the bridge, from many experiments, is not more than $2\frac{1}{4}$ ft. per second.

Here then $b = 994$, $c = 820$, $v = 2\frac{1}{4} = \frac{9}{4}$.

Hence $\frac{1\cdot42b^2 - c^2}{64c^2} = \frac{1403011 - 672400}{64 \times 672400} = \cdot01722$.

And $v^2 = \frac{81}{16} = 5\frac{1}{16}$.

Theref. $\cdot01722 \times 5\frac{1}{16} = \cdot0872$ ft. $= 1$ in. the fall required; which is about half an inch more than the greatest fall observed by Mr. Labelye.

And, for Blackfriar's-bridge, the fall will be much the same as that of Westminster.

FINIS.

T. Davison, Lombard-street,
Whitefriars, London.